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Asymptotic behaviour of first passage time distributions for Lévy processes

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Abstract

Let X be a real valued Lévy process that is in the domain of attraction of a stable law without centering with norming function c. As an analogue of the random walk results in [19] and [8] we study the local behaviour of the distribution of the lifetime ζ under the characteristic measure $\underline{n}$ of excursions away from 0 of the process X reflected in its past infimum, and of the first passage time of X below 0, $T_0 = \inf\{t > 0 : X_t < 0\}$, under $\mathbb{P}_x(\cdot)$, for $x > 0$, in two different regimes for x, viz. $x = o(c(\cdot))$ and $x > Dc(\cdot)$, for some $D > 0$. We sharpen our estimates by distinguishing between two types of path behaviour, viz. continuous passage at T_0 and discontinuous passage. In the way to prove our main results we establish some sharp local estimates for the entrance law of the excursion process associated to X reflected in its past infimum.

Keywords: Lévy processes, first passage time distribution, local limit theorems, fluctuation theory.

Mathematics subject classification: 60G51, 60 G52, 60F99

1 Introduction and main results

Let X be a real valued Lévy process with law $\mathbb{P}$ and characteristics (a, σ, Π). We are interested in the **local** behaviour of the distribution of the first passage time of X below 0, i.e. $T_0 = \inf\{t > 0 : X_t < 0\}$, under $\mathbb{P}_x(\cdot)$, for $x > 0$. We start by investigating the existence of a density for this distribution, but our main focus is on the asymptotic behaviour of this density, or when it fails to exist, other local-limit type results, all of which are analogues of results for random walks in [8]. We will assume throughout that under $\mathbb{P}$ neither X nor $-X$ is a subordinator; in the first case the problem has no sense, and in the second case a different approach

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in needed as our methods rely on the possibility of excursions above the minimum. In addition, since the results for compound Poison processes can be deduced directly from the random walk results in [8], we will also assume that $\Pi(\mathbb{R}) = \infty$. If additionally 0 is regular for the half-line $(-\infty, 0)$ under $\mathbb{P}$, (we abbreviate this to "X is regular downwards") then $T_0 \equiv 0$ under $\mathbb{P}_0$, so as an analogue of the random walk results in [19] we study the distribution of the lifetime $\zeta = \inf\{t > 0 : \epsilon_t = 0\}$ under the characteristic measure $\underline{n}$ of excursions away from 0 of the process reflected in its infimum.

It turns out that we need to distinguish between two types of path behaviour, viz continuous passage at T_0 and discontinuous passage. It is known that the first only has positive probability if X "creeps downwards" under $\mathbb{P}$, or equivalently the drift d^* of the downgoing ladder height process is positive. We start by showing that, on the event of discontinuous passage, T_0 admits a density under $\mathbb{P}_x, x > 0$, and a similar result holds in the excursion case. In particular, the first passage time distribution is absolutely continuous in the case $d^* = 0$. However, when $d^* > 0$, it can happen that on the event of continuous passage the distribution of T_0 is singular with respect to Lebesgue measure. We therefore need to formulate our results differently in these two situations, and the proofs are also somewhat different.

For the asymptotic results which are the main topic of this paper, we assume that X is in the domain of attraction of a stable distribution without centering, that is there exists a deterministic function $c : (0, \infty) \to (0, \infty)$ such that

$$\frac{X_t}{c(t)} \xrightarrow{\mathcal{D}} Y(1), \quad \text{as } t \to \infty, \tag{1}$$

with $Y(1)$ a strictly stable random variable of parameter $0 < \alpha \le 2$, and positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$. In this case we will use the notation $X \in D(\alpha, \rho)$, and put $\overline{\rho} = 1 - \rho$. Hereafter $(Y_t, t \ge 0)$ will denote an α-stable Lévy process with positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$.

It is well known that in this case the function c is regularly varying at infinity with index $1/\alpha$. **Throughout this paper we will use the notation $\eta = 1/\alpha$.**

It is also known that the bivariate downgoing ladder process (τ^*, H^*) is in the domain of attraction of a bivariate $(\overline{\rho}, \alpha\overline{\rho})$ stable law, and since $\overline{\rho}(t) = \mathbb{P}(X_t < 0) \to \overline{\rho}$, it follows from Spitzer's formula that

$$\underline{n}(\zeta > \cdot) \in RV(-\overline{\rho}), \tag{2}$$

where $RV(\beta)$ denotes the class of functions which are regularly varying with index β at ∞. Our first concern is to obtain a local version of (2), but we need to consider separately the contributions coming from continuous and discontinuous passage. So let $\nu(x) = \mathbb{P}_x(C_0)$ where

$$C_0 = \{X(T_0-) = 0\}$$

is the event of crossing level 0 continuously. Then it is known that many processes have $\nu(x) \equiv 0$, e.g. any stable process which is not spectrally positive, and spectrally positive processes have $\nu(x) \equiv 1$, provided they do not drift to ∞. The case $0 < \nu(x) < 1$, $x > 0$ arises if and only if H^*

has a positive drift d^* and in this case there is a renewal density u^*, and $\nu(x) = d^*u^*(x)$, so that $\lim_{x\downarrow 0}\nu(x) = 1$, and

$$\lim_{x\to\infty}\nu(x) = \frac{d^*}{m^*} := q \in [0,1)$$

where $m^* = \mathbb{E}H_1^* = d^* + \int_0^\infty \overline{\mu}^*(x)dx$. (See [4].) However, when $X \in D(\alpha,\rho)$, $\mathbb{E}H_1^* = \infty$ whenever $\alpha\overline{\rho} < 1$; when $\alpha\overline{\rho} = 1$, $\mathbb{E}H_1^*$ can be finite or infinite, even though the limiting stable process is spectrally positive. So, except in this special case, $q = 0$ and possibly $\mathbb{P}_x(C_0)$ should be negligible.

The following result verifies this intuition, since, except in this special case, because $c(\cdot) \in RV(\eta)$ and $\underline{n}(\zeta > \cdot) \in RV(-\overline{\rho})$, we have $1/c(t) = o(\underline{n}(\zeta > t))$. In it we write $p = 1-q$, f for the density of Y_1 and $\underline{n}^c(t,\Delta]$ and $\underline{n}^d(t,\Delta]$ for $\underline{n}(\zeta \in (t,t+\Delta], \epsilon(\zeta-) = 0)$ and $\underline{n}(\zeta \in (t,t+\Delta], \epsilon(\zeta-) > 0)$, respectively. The quantity $f(0)$ plays an important role in our estimates, known expressions for it can be found in [21] equation (2.2.11) or in [17] equations (14.30-33).

Theorem 1 *Suppose $X \in D(\alpha,\rho)$ and fix any $\Delta_0 > 0$: then uniformly for $\Delta \in (0,\Delta_0]$*

$$\lim_{t\to\infty}\frac{t\underline{n}(\zeta \in (t,t+\Delta])}{\Delta\underline{n}(\zeta > t)} = \overline{\rho}. \tag{3}$$

More precisely, we have,

(i) Whenever $\Pi((-\infty,0)) > 0$, $\exists h_0$ such that $\underline{n}^d(t,\Delta] := \int_t^{t+\Delta} h_0(s)ds$, and

$$\lim_{t\to\infty}\frac{th_0(t)}{\overline{\rho}\underline{n}(\zeta > t)} = p. \tag{4}$$

(ii) When $d^* > 0$,

$$\lim_{t\to\infty}\frac{tc(t)\underline{n}^c(t,\Delta]}{\overline{\rho}\Delta} = f(0)d^*, \text{ uniformly for } \Delta \in (0,\Delta_0], \tag{5}$$

and in particular, if also $\alpha\overline{\rho} = 1$,

$$\lim_{t\to\infty}\frac{\underline{n}^c(t,\Delta]}{\overline{\rho}\Delta\underline{n}(\zeta > t)} = q. \tag{6}$$

(iii) If $\Pi((-\infty,0)) = 0$, then $\underline{n}^d(t,\Delta] \equiv 0$, $q = 1$, and

$$\lim_{t\to\infty}\frac{t\underline{n}^c(t,\Delta]}{\Delta\overline{\rho}\underline{n}(\zeta > t)} = 1, \text{ uniformly for } \Delta \in (0,\Delta_0]. \tag{7}$$

For the case $x > 0$, we state here only the analogue of (3), but in the sequel we will also state and prove results analogous to (4), (5), (6) and (7).

We write U^* for the renewal measure of H^* and $\tilde{h}_x(\cdot)$ for the density of the first passage time to $(-\infty,0)$ of Y starting from $x > 0$.

Remark 2 *From now on, the phrase "uniformly in Δ" will be used as an abbreviation for "uniformly in $\Delta \in (0,\Delta_0]$ for any fixed $\Delta_0 > 0$".*

Theorem 3 *Uniformly in Δ and $x > 0$*

$$\frac{t\mathbb{P}_x(T_0 \in (t, t+\Delta])}{\Delta} = \tilde{h}_{x_t}(1) + o(1) \text{ as } t \to \infty. \tag{8}$$

Also, uniformly in Δ and $x > 0$ such that $x_t := x/c(t) \to 0$,

$$\lim_{t\to\infty} \frac{t\mathbb{P}_x(T_0 \in (t, t+\Delta])}{\Delta U^*(x)\underline{n}(\zeta > t)} = \overline{\rho}. \tag{9}$$

Remark 4 *If $x_t \to 0$ or ∞, $\tilde{h}_{x_t}(1) \to 0$ and the RHS of (8) is $o(1)$, so it is sufficient to show that for any $D > 1$ (8) holds uniformly in Δ and x such that $x_t \in [D^{-1}, D]$.*

Remark 5 *In Lemma 14 in the next section we will see that if $x_t \to 0$ then $U^*(x)\underline{n}(\zeta > t) \to 0$ and hence the estimate (9) is more precise than (8) in the "small x_t" situation.*

We finish this section by stating two propositions which play an important part in the proof of each of the above results.

Write U for the renewal measure in the upgoing ladder height process H, f for the density of Y_1, and g for the probability density function of the (α, ρ)-stable meander of length 1 at time 1.

Proposition 6 *Assume that $X \in D(\alpha, \rho)$. Then uniformly in Δ and $y \geq 0$ such that $y_t := y/c(t) \to 0$,*

$$tc(t)\underline{n}\left(\epsilon_t \in (y, y+\Delta), \zeta > t\right) \backsim f(0) \int_y^{y+\Delta} U(z)dz \text{ as } t \to \infty. \tag{10}$$

Proposition 7 *Assume that $X \in D(\alpha, \rho)$. Then uniformly in Δ and $y \geq 0$,*

$$c(t)\underline{n}\left(\epsilon_t \in (y, y+\Delta) | \zeta > t\right) = \Delta\left(g\left(y_t\right) + o(1)\right) \text{ as } t \to \infty. \tag{11}$$

Remark 8 *If we made the simplifying assumptions that $\alpha\overline{\rho} < 1$, that X is regular upwards and downwards and does not creep downwards, the following proofs would be considerably simplified. However we believe the additional work is justified because it would be unnatural to exclude the case $\alpha\overline{\rho} = 1$, or to make any assumptions about the local behaviour of X.*

2 Preliminaries

We recall a few customary notations in fluctuation theory. For background about fluctuation theory for Lévy processes the reader is referred to the books [3], [9], and [14].

The process $X_t - I_t = X_t - \inf_{0\leq s\leq t} X_s$, $t \geq 0$ is a strong Markov process, the point process of its excursions out of zero forms a Poisson point process with intensity or excursion measure $\underline{n}$. We will denote by $\{\epsilon_t, t > 0\}$ the generic excursion process and by ζ its lifetime. It is known that under $\underline{n}$ the excursion process is Markovian with semigroup given by

$\mathbb{P}_x(X_t \in dy, t < T_0)$. We will denote by L^* the local time at 0 for $X - \underline{X}$, and we will assume WLOG that it is normalized so that

$$\mathbb{E}\left(\int_0^\infty e^{-s} dL_s^*\right) = 1. \tag{12}$$

We will denote by τ^* the right continuous inverse of the local time L^*, and refer to it as the downward ladder time process, and call $\{H_t^* = -X_{\tau_t^*}, t \geq 0\}$ the downward ladder height process. The potential measure of the bivariate process (τ^*, H^*) will be denoted by

$$W^*(dt, dx) = \int_0^\infty ds \mathbb{P}(\tau_s^* \in dt, H_s^* \in dx), \qquad t \geq 0, x \geq 0.$$

The marginal in space of W^* is the potential measure of the downward ladder height process H^*, and we will denote by U^* its associated renewal function

$$U^*(a) := W^*([0,\infty) \times [0,a]) = \int_0^\infty ds \mathbb{P}(H_s^* \leq a), \qquad a \geq 0.$$

Analogously, the function V^* will denote the renewal function of the downward ladder time process, τ^*. We will use a similar notation for the analogous objects defined in terms of X^* but we will remove the symbol $*$ from them, and the excursion measure will be denoted by $\overline{n}$.

An important duality relation, which we will use extensively, connects W^* and W with the characteristic measures $\underline{n}$ and $\overline{n}$: see Lemma 1 in [6].

Lemma 9 *Let a, a^* denote the drifts in the ladder time processes τ and τ^* : then on $[0,\infty) \times [0,\infty)$ we have the identities*

$$\begin{aligned} W(dt, dx) &= a^* \delta_{\{(0,0)\}}(dt, dx) + \underline{n}(\epsilon_t \in dx, \zeta > t) dt, & (13) \\ W^*(dt, dx) &= a \delta_{\{(0,0)\}}(dt, dx) + \overline{n}(\epsilon_t \in dx, \zeta > t) dt, & (14) \end{aligned}$$

so that, in particular

$$\begin{aligned} U(x) &= a^* + \int_0^\infty \int_0^x \underline{n}(\epsilon_t \in dy, \zeta > t) dt, & (15) \\ U^*(x) &= a + \int_0^\infty \int_0^x \overline{n}(\epsilon_t \in dy, \zeta > t) dt. & (16) \end{aligned}$$

Remark 10 *Note that $a = 0$ (respectively $a^* = 0$) is equivalent to X being regular downwards (respectively upwards), and since we exclude the Compound Poisson case, at most one of a and a^* is positive.*

We write $\mathbb{P}^*$ for the law of the dual Lévy process $X^* = -X$, and define

$$\overline{\Pi}(y) = \Pi(y, \infty), \quad \overline{\Pi}^*(y) := \Pi(-\infty, -y), \quad y \geq 0.$$

Put

$$\begin{aligned} h_x(t) &= \mathbb{E}_x\left(\overline{\Pi}^*(X_t), t < T_0\right) = \int_0^\infty \mathbb{P}_x(X_t \in dy, t < T_0)\overline{\Pi}^*(y), \ x, t > 0, \\ h_0(t) &= \underline{n}(\overline{\Pi}^*(\epsilon_t), t < \zeta) = \int_0^\infty \underline{n}(\epsilon_t \in dy, t < \zeta)\overline{\Pi}^*(y), \ t > 0, \end{aligned}$$

Lemma 11 *We have the following formulae*

$$\mathbb{P}_x\left(T_0 \in dt, X_{T_0-} > 0\right) = h_x(t)dt, \;\; x, t > 0; \tag{17}$$

$$\underline{n}(\zeta \in dt, \epsilon_{\zeta-} > 0) = h_0(t)dt, \;\; t > 0. \tag{18}$$

In particular, for $x > 0$, we have that if X does not creep downward then the law of T_0 under $\mathbb{P}_x$ is absolutely continuous w.r.t Lebesgue measure.

Proof. Let f be a measurable and bounded function. Using the fact that the jumps of X form a Poisson point process in $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure $dt\Pi(dz)$, and the compensation formula, we get

$$\begin{aligned}
\mathbb{E}_x(f(T_0)1_{\{X_{T_0-}>0\}}) &= \mathbb{E}\left(\sum_{s>0} f(s)1_{\{I_{s-}>-x, \Delta X_s < -(x+X_{s-})\}}\right) \\
&= \mathbb{E}\left(\int_0^\infty ds f(s) 1_{\{I_{s-}>-x\}} \Pi\left(-\infty, -(x+X_{s-})\right)\right) \\
&= \mathbb{E}\left(\int_0^\infty ds f(s) 1_{\{I_{s}>-x\}} \Pi\left(-\infty, -(x+X_{s})\right)\right) \\
&= \mathbb{E}\left(\int_0^\infty ds f(s) 1_{\{s<T_{-x}\}} \Pi\left(-\infty, -(x+X_{s})\right)\right) \\
&= \int_0^\infty ds f(s) \mathbb{E}\left(\Pi\left(-\infty, -(x+X_s)\right) 1_{\{s<T_{-x}\}}\right) \\
&= \int_0^\infty ds f(s) \mathbb{E}_x\left(\Pi\left(-\infty, -X_s\right) 1_{\{s<T_0\}}\right).
\end{aligned}$$

We next prove the identity under $\underline{n}$. For $t, s \geq 0$, we have from the Markov property under $\underline{n}$ that

$$\begin{aligned}
\underline{n}(\zeta > t+s, \epsilon_{\zeta-} > 0) &= \underline{n}\left(\mathbb{P}_{\epsilon_s}(T_0 > t, X_{T_0-} > 0), s < \zeta\right) \\
&= \int_{[0,\infty)} \underline{n}(\epsilon_s \in dy, s < \zeta) \int_t^\infty du \mathbb{E}_y(\overline{\overline{\Pi}}^-(X_u), u < T_0) \\
&= \int_t^\infty du \int_{[0,\infty)} \underline{n}(\epsilon_s \in dy, s < \zeta) \mathbb{E}_y(\overline{\overline{\Pi}}^-(X_u), u < T_0) \\
&= \int_t^\infty du \underline{n}\left(\overline{\overline{\Pi}}^-(\epsilon_{u+s}), u+s < \zeta\right) \\
&= \int_{t+s}^\infty dv \underline{n}\left(\overline{\overline{\Pi}}^-(\epsilon_v), v < \zeta\right).
\end{aligned}$$

■

In the case where the process creeps downward there is no general result about the absolute continuity of the law of T_0 under $\mathbb{P}_x(\cdot | X_{T_0} = 0)$. However, if X is spectrally positive then the downward ladder time process is in fact the first passage time process $(T_{-x}, x > 0)$, where

$$T_{-x} = \inf\{t > 0 : X_t < -x\},$$

and T_0 under $\mathbb{P}_x$ has the same law as T_{-x} under $\mathbb{P}$. By the continuous time version of the "Ballot Theorem" (see Corollary 3, p190 of [3]) it follows that the law of T_0 is absolutely continuous under $\mathbb{P}_x$ for all $x > 0$ iff the $\mathbb{P}_x$

distribution of X_t is absolutely continuous for all $t > 0$. But Orey [15] gave examples where this fails. In these examples, X has infinite variation, but it is also easy to see that for instance if $X_t = N_t - t$, $t \geq 0$, where N_t is a Poisson process with parameter $\lambda > 0$, the law of T_0 under $\mathbb{P}_x$ is atomic with support in $\{x + n, n \in \mathbb{Z}^+\}$. (This example is not really relevant to the sequel, because such a process cannot be in the domain of attraction of any stable law.)

In order to shorten the notation throughout the rest of the paper we will understand the following terms as equal, for $s > 0$,

$$\underline{n}_s(dy) = \underline{n}(\epsilon_s \in dy) = \underline{n}(\epsilon_s \in dy, s < \zeta), \qquad y > 0.$$

Since in any case we will be integrating over $(0, \infty)$ there will not be any risk of confusion. Analogous notation will be used under the excursion measure $\overline{n}$.

Throughout this paper we will make systematic use of the identities in the following Lemma 12 as well as the estimates in the Lemma 14.

Lemma 12 *(i) The semigroup of X can be expressed as: for $x, y \in \mathbb{R}$*

$$\begin{aligned}\mathbb{P}_x\left(X_t \in dy\right) = &\int_{s=0}^{t} ds \int_{z>(x-y)^+} \overline{n}_s(dz)\underline{n}_{t-s}(dy + z - x) \\ &+ a\underline{n}_t(dy - x)\mathbf{1}_{\{y \geq x\}} + a^*\overline{n}_t(x - dy)\mathbf{1}_{\{y \leq x\}}.\end{aligned} \tag{19}$$

(ii) The semigroup of X killed at its first entrance into $(-\infty, 0)$ can be expressed as: for $x, y \in \mathbb{R}^+$

$$\begin{aligned}\mathbb{P}_x\left(X_t \in dy, t < T_0\right) = &\int_{s=0}^{t} ds \int_{z \in ((x-y)^+, x]} \overline{n}_s(dz)\underline{n}_{t-s}\left(dy + z - x\right) \\ &+ a\underline{n}_t(dy - x)\mathbf{1}_{\{y \geq x\}} + a^*\overline{n}_t(x - dy)\mathbf{1}_{\{y \leq x\}}.\end{aligned} \tag{20}$$

(iii) The one-dimensional distribution of the excursion process under $\underline{n}$ can be decomposed as: for $x > 0$

$$t\underline{n}(\epsilon_t \in dx) = \int_0^t ds \int_{z \in [0,x]} \underline{n}_s(dz)\mathbb{P}_z(X_{t-s} \in dx) + a^*\mathbb{P}(X_t \in dx). \tag{21}$$

Proof. The identity (21) is due to Alili and Chaumont [1], and it is a generalization of a result for random walks due to Alili and Doney [2], see also [11] for further details about the proof of this result. The proof of the identities (19) and (20), together with other useful fluctuation identities can be found in [6]. ■

In what follows, $k, k_1, k_2, \cdots$ will denote fixed positive constants whereas C will denote a generic constant whose value can change from line to line. As previously remarked, the norming function $c(\cdot) \in RV(\eta)$,where $\eta = 1/\alpha$. More precisely we will assume, with no loss of generality, that Y is a standard stable process, and c can be taken to be a continuous, monotone increasing inverse of the quantity $x^2/m(x)$; where $m(x) = \int_{-x}^{x} y^2\Pi(dy)$ and necessarily $m(\cdot) \in RV(2 - \alpha)$. It follows from

this that, when $\alpha < 2$, we have $t\overline{\Pi}(c(t)) \to k$ and $t\overline{\Pi}^*(c(t)) \to k^*$, with $k^* > 0$ if $\alpha\overline{\rho} < 1$, and $k^* = 0$ if $\alpha\overline{\rho} = 1$, when necessarily $k > 0$. Finally when $\alpha = 2$, we have $t(\overline{\Pi}(c(t)) + \overline{\Pi}^*(c(t))) \to 0$, so we can take $k = k^* = 0$.

The following local limit theorem is a crucial tool.

Proposition 13 *Assume that $X \in D(\alpha, \rho)$, with $\alpha\overline{\rho} \le 1$. Then uniformly in Δ and $x \in \mathbb{R}$,*

$$c(t)\mathbb{P}(X_t \in (x, x + \Delta]) = \Delta(f(\frac{x}{c(t)}) + o(1)) \text{ as } t \to \infty. \tag{22}$$

Consequently given any $\Delta_0 > 0$ there are constants k_0 and t_0 such that

$$c(t)\mathbb{P}(X_t \in (x, x + \Delta]) \le k_0\Delta \text{ for all } t \ge t_0 \text{ and } \Delta \in (0, \Delta_0]. \tag{23}$$

We have not been able to locate this result in the literature, but it can easily be proved by repeating the argument used for non-lattice random walks in [18], with very minor changes.

Other useful facts are in:

Lemma 14 *Assume that $X \in D(\alpha, \rho)$, with $\alpha\overline{\rho} \le 1$. We have that*

$$U^*(c(t)) \sim \frac{k_1}{\underline{n}(\zeta > t)}, \qquad U(c(t)) \sim \frac{k_2}{\overline{n}(\zeta > t)} \qquad t \to \infty, \tag{24}$$

Also

$$t\underline{n}(\zeta > t)\overline{n}(\zeta > t) \xrightarrow[t\to\infty]{} k_3, \tag{25}$$

where $k_3 = (\Gamma(\rho)\Gamma(\overline{\rho}))^{-1}$.

Proof. Let Y^t the Lévy process defined by $Y^t_s := \frac{X_{ts}}{c(t)}$, $s \ge 0$, so that Y^t converges weakly to Y. By a recent result by Chaumont and Doney [7], see also [20] chapter 3, Lemma 3.4.2, we know that the ladder processes associated to Y^t also converge weakly towards those associated to Y. Hence the upward ladder height subordinator associated to Y^t converges to a stable subordinator of parameter $\alpha\rho$, where if $\alpha\rho = 1$ we interpret the limit as a pure drift. On the one hand, when we write this in terms of Laplace exponents we get

$$\lim_{t\to\infty} \kappa^{(t)}(0, \lambda) = B\lambda^{\alpha\rho}, \qquad \lambda \ge 0,$$

where $\kappa^{(t)}(\cdot, \cdot)$ denotes the Laplace exponent of the upward ladder process associated to Y^t, and B is a constant depending on the normalization of the local time, which because of the normalization chosen here equals 1. On the other hand, when we write Fristedt's formula for $\kappa^{(t)}$ we get the

identities

$$
\begin{aligned}
\kappa^{(t)}(0,\lambda) &= \exp\left\{\int_0^\infty \frac{ds}{s}\int_{[0,\infty)}\left(e^{-s}-e^{-\lambda x}\right)\mathbb{P}(Y_s^t\in dx)\right\}\\
&= \exp\left\{\int_0^\infty \frac{ds}{s}\int_{[0,\infty)}\left(e^{-s/t}-e^{-\lambda x/c(t)}\right)\mathbb{P}(X_s\in dx)\right\}\\
&= \exp\left\{\int_0^\infty \frac{ds}{s}\int_{[0,\infty)}\left(e^{-s}-e^{-\lambda x/c(t)}\right)\mathbb{P}(X_s\in dx)\right\}\\
&\quad\times\exp\left\{-\int_0^\infty \frac{ds}{s}\left(e^{-s}-e^{-s/t}\right)\mathbb{P}(X_s\geq 0)\right\}\\
&= \frac{\kappa(0,\lambda/c(t))}{\kappa(1/t,0)},
\end{aligned}
$$

for $\lambda\geq 0, t>0$; where $\kappa(\cdot,\cdot)$ denotes the Laplace exponent of the upward ladder process (τ,H). In particular, since $\kappa(0,1)=1$,

$$\kappa(0,1/c(t))\sim\kappa(1/t,0),\qquad \text{as } t\to\infty.$$

By the hypothesis of the Lemma we have $\kappa(\cdot,0)\in RV(\rho)$ and $\kappa(0,\cdot)\in RV(\alpha\rho)$. To conclude we use Proposition III.1 in [3] to deduce that

$$\kappa(1/t,0)\sim\Gamma(1-\rho)\overline{n}(\zeta>t),\qquad \kappa(0,1/t)\sim\frac{1}{\Gamma(1+\alpha\rho)U(t)},\qquad \text{as } t\to\infty.$$

It follows that

$$U(c(t))\sim\frac{1}{\Gamma(1+\alpha\rho)\Gamma(1-\rho)\overline{n}(\zeta>t)},\qquad t\to\infty.$$

By applying this result to the dual Lévy process $-X$ we get the first asymptotic.

To prove (25) we observe that from Lemma 9

$$V[0,t)=a+\int_0^t \underline{n}(\zeta>s)ds,\qquad t\geq 0.$$

Applying again Proposition III.1 in [3] but this time to the upward ladder time subordinator we get that

$$V[0,t)\sim\frac{1}{\Gamma(1+\rho)\Gamma(1-\rho)\overline{n}(\zeta>t)},\qquad t\to\infty.$$

Then by Karamata's theorem we have also that

$$\int_0^t \underline{n}(\zeta>s)ds\sim\frac{1}{1-\overline{\rho}}t\underline{n}(\zeta>t),\qquad t\to\infty.$$

The result follows by equating the terms. ■

A consequence of the fact that $(X(ts)/c(t), s\geq 0)$ converges in law to $(Y(s), s\geq 0)$, is that

Lemma 15 *Assume that $X \in D(\alpha,\rho)$, with $\alpha\overline{\rho} \leq 1$. Then as $t \to \infty$*

$$\underline{n}(\epsilon_t \in c(t)dx|\zeta > t) \xrightarrow{D} \mathbb{P}(Z_1 \in dx),$$

where Z_1 denotes the stable meander of length 1 at time 1 based on Y.

Proof. Let $\mathbb{P}^{\uparrow}$ denote the law of "X conditioned to stay positive, starting from zero". (For a proper definition of this see e.g. Chapter 8 of [9].) Then, using the absolute continuity between $\underline{n}$ and $\mathbb{P}^{\uparrow}$, and Lemma 14, we have that over compact sets in $(0,\infty)$

$$\begin{aligned} \underline{n}(\epsilon_t \in c(t)dx|\zeta > t) &= \frac{C\mathbb{P}^{\uparrow}(X_t \in c(t)dx)}{\underline{n}(\zeta > t)U^*(c(t)x)} \\ &\backsim \frac{C\mathbb{P}^{\uparrow}(X_t \in c(t)dx)}{x^{\alpha\overline{\rho}}\underline{n}(\zeta > t)U^*(c(t))} \backsim \frac{C\mathbb{P}^{\uparrow}(X_t \in c(t)dx)}{x^{\alpha\overline{\rho}}} \\ &\to Cx^{-\alpha\overline{\rho}}\mathbb{P}^{\uparrow}(Y_1 \in dx) = C\underline{n}^Y(\epsilon(1) \in dx|\zeta > 1) \\ &= C\mathbb{P}(Z_1 \in dx). \end{aligned}$$

Here the convergence of $\mathbb{P}^{\uparrow}(X_t \in c(t)dx)$ to $\mathbb{P}^{\uparrow}(Y_1 \in dx)$ is a consequence of results in [7]. The above argument is valid over compact sets of $(0,\infty)$, thus proving the vague convergence. To get the convergence in distribution we should also verify that the mass is preserved, but this is straightforward from the fact that $\underline{n}(\epsilon_t \in (0,\infty)|\zeta > t) = 1 = \mathbb{P}(Z_1 \in (0,\infty))$. This would finish the proof if we can guarantee that $C = 1$, but this is a consequence of the normalization chosen. ■

3 Proof of Propositions 6 and 7

We start by proving the following Lemmas.

Lemma 16 *Put $\kappa_t^{\Delta}(x) = \underline{n}^c(t,\Delta] + \underline{n}^d(t,\Delta] = \underline{n}(\epsilon_t \in (x, x+\Delta])$ and fix $\Delta_0 > 0$. Then, for all values of $\alpha\overline{\rho}$, for some constants k_4 and t_0 we have, uniformly for $0 < \Delta \leq \Delta_0$ and $0 \leq x \leq c(t)$,*

$$tc(t)\kappa_t^{\Delta}(x) \leq k_4\Delta U(x+\Delta) \text{ for } t \geq t_0. \tag{26}$$

Proof. This is similar to the proof of Lemma 20 in [19]. First we use the bound (23) from Proposition 13 to get

$$\begin{aligned} c(t)\kappa_t^{\Delta}(x) &= c(t)\int_{y>0} \underline{n}(\epsilon(t/2) \in dy)\mathbb{P}_y(X_{t/2} \in (x, x+\Delta], T_0 > t/2) \\ &\leq \frac{k_0\Delta c(t)}{c(t/2)}\int_{y>0} \underline{n}(\epsilon(t/2) \in dy) = \frac{k_0\Delta c(t)\underline{n}(\zeta > t/2)}{c(t/2)} \\ &\leq k_5\Delta\underline{n}(\zeta > t). \end{aligned} \tag{27}$$

Next, it is immediate from equation (21) that

$$t\kappa_t^{\Delta}(x) = \int_0^t du\int_{z=x}^{x+\Delta}\int_{y=0}^{z} \mathbb{P}(X_{t-u} \in dy)\underline{n}_u(dz-y) + a^*\mathbb{P}(X_t \in (x, x+\Delta]). \tag{28}$$

It is useful to note that we can write the inner double integral either as

$$\int_{y=0}^{x+\Delta} \mathbb{P}(X_{t-u} \in dy) \underline{n}_u([(x-y)^+, x-y+\Delta)),$$

or as

$$\begin{aligned} & \int_{w=0}^{x+\Delta} \underline{n}_u(dw) \int_{y=(x-w)^+}^{x-w+\Delta} \mathbb{P}(X_{t-u} \in dy) \\ = & \int_{w=0}^{x+\Delta} \underline{n}_u(dw) \mathbb{P}(X_{t-u} \in [(x-w)^+, x-w+\Delta)). \end{aligned}$$

So we take $\delta \in (0,1)$ and write $t\kappa_t^{\Delta}(x) = J_1^{\delta} + J_2^{\delta} + a^*\mathbb{P}(X_t \in (x, x+\Delta])$, where

$$\begin{aligned} J_1^{\delta} &= \int_0^{\delta t} du \int_{w=0}^{x+\Delta} \underline{n}_u(dw) \mathbb{P}(X_{t-u} \in [(x-w)^+, x-w+\Delta)), \\ J_2^{\delta} &= \int_{\delta t}^{t} du \int_{y=0}^{x+\Delta} \mathbb{P}(X_{t-u} \in dy) \underline{n}_u([(x-y)^+, x-y+\Delta)), \text{ and} \end{aligned}$$

$$a^*\mathbb{P}(X_t \in (x, x+\Delta]) = \frac{a^*\Delta}{c(t)} \{f(x/c(t)) + o(1)\}, \tag{29}$$

where we have used (22). We see immediately from (27) that

$$J_2^{\delta} \leq \frac{k_5 \Delta \underline{n}(\zeta > \delta t)}{c(\delta t)} \int_0^{(1-\delta)t} \mathbb{P}(0 < X_u \leq x+\Delta) du$$

From (19) and the subadditivity of U we have, for $y > 0$

$$\begin{aligned} \int_0^{(1-\delta)t} \mathbb{P}(0 < X_u \leq y) du &= \int_0^{(1-\delta)t} du \int_{s=0}^{u} ds \int_{z=0}^{\infty} \overline{n}_s(dz) \underline{n}_{u-s}([(z-y)^+, z]) \\ &= \int_{s=0}^{(1-\delta)t} ds \int_{v=0}^{(1-\delta)t-s} dv \int_{z=0}^{\infty} \overline{n}_s(dz) \underline{n}_v([(z-y)^+, z]) \\ &\leq \int_0^{(1-\delta)t} ds \int_{z=0}^{\infty} \overline{n}_s(dz) [U(z) - U((z-y)^+)] \\ &\leq U(y) \int_0^{(1-\delta)t} ds \overline{n}(\zeta > s) \backsim \frac{U(y)(1-\delta)^{\overline{\rho}} t \overline{n}(\zeta > t)}{\overline{\rho}}, \end{aligned}$$

and using this with $y = x + \Delta$ and (25) gives

$$\limsup_{t \to \infty} \frac{c(t) J_2}{\Delta U(x+\Delta)} \leq \frac{k_5 k_3 (1-\delta)^{\overline{\rho}}}{\overline{\rho} \delta^{\overline{\rho}+\eta}}. \tag{30}$$

For the other term, we again use the bound (23) to get

$$\begin{aligned} J_1^{\delta} &\leq k_0 \Delta \int_0^{t\delta} \frac{du}{c(t-u)} \int_{w=0}^{x+\Delta} \underline{n}_u(dw) \\ &\leq \frac{k_0 \Delta (U(x+\Delta) - a^*)}{c((1-\delta)t)} \backsim \frac{k_0 \Delta (U(x+\Delta) - a^*)}{(1-\delta)^{\eta} c(t)}. \end{aligned} \tag{31}$$

Choosing $\delta = 1/2$, the result follows from (29), (30) and (31). ■

Corollary 17 *The bound (26), with a suitable k, holds uniformly in $x \geq 0$.*

Proof. Just note that, by (27) $tc(t)\kappa_t^\Delta(x) \leq k_5 \Delta t \underline{n}(\zeta > t) \backsim k_3 k_5 \Delta / \overline{n}(\zeta > t)$ and if $x \geq c(t)$ we have $U(x+\Delta) \geq U(c(t))k_2/\overline{n}(\zeta > t)$. ■

We can now prove Proposition 6, which we restate as

Proposition 18 *Uniformly in Δ and uniformly as $x/c(t) \to 0$,*

$$tc(t)\kappa_t^\Delta(x) \backsim f(0)\int_x^{x+\Delta} U(y)dy =: f(0)U^\Delta(x) \text{ as } t \to \infty.$$

Proof. We again use the representation $t\kappa_t^\Delta(x) = J_1^\delta + J_2^\delta + a^*\mathbb{P}(X_t \in (x, x+\Delta])$, but this time we will be choosing δ small. Recall that the behaviour of the final term here is given by (29). Using Proposition 13, we get that as $t \to \infty$, uniformly in Δ and δ,

$$\begin{aligned} J_1^\delta &\sim \int_0^{\delta t} \frac{f(0)}{c(t-u)} du \int_{z=0}^{x+\Delta} \underline{n}_u(dz)(x - z + \Delta - (x-z)^+) \\ &= \int_0^{\delta t} \frac{f(0)}{c(t-u)} du \int_{z>0}^{x+\Delta} (x - z + \Delta - (x-z)^+) W(du, dz). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} &\int_{z>0}^{x+\Delta} (x - z + \Delta - (x-z)^+) W(du, dz) \\ =\ &\Delta W(du, (0,x]) + \int_{y=0}^{\Delta} (\Delta - y) W(du, x+dy) \\ =\ &\int_{y=0}^{\Delta} W(du, (0, x+\Delta-y))dy, \end{aligned}$$

and we see that J_1^δ is asymptotically bounded below by

$$\frac{f(0)}{c(t)}\left(U^\Delta(x) - \Delta a^* - \int_{\delta t}^\infty \kappa_u^\Delta(x)du\right).$$

The same argument gives the asymptotic upper bound of

$$\frac{f(0)}{c(t(1-\delta))}\left(U^\Delta(x) - \Delta a^* - \int_{\delta t}^\infty \kappa_u^\Delta(x)du\right).$$

By Corollary 17, for each fixed $\delta > 0$ we have the asymptotic bound

$$\begin{aligned} \int_{\delta t}^\infty \kappa_u^\Delta(x)du &\leq k_4 \Delta U(x+\Delta) \int_{\delta t}^\infty du/(uc(u)) \\ &\backsim C(\delta)\Delta U(x+\Delta)/c(t) \\ &= o(1)\Delta U(x+\Delta) = o(1)U^\Delta(x), \end{aligned}$$

where we observe that Erickson's [12] bounds give

$$\frac{\Delta U(x+\Delta)}{U^\Delta(x)} \leq \left(\begin{array}{ll} \frac{\Delta U(x+\Delta)}{\Delta U(x)} \leq C & \text{for } \Delta \leq x, \\ \frac{\Delta U(2\Delta)}{\Delta/2 U(\Delta/2)} \leq C & \text{for } x \leq \Delta. \end{array} \right.$$

Hence

$$c(t)J_1^\delta + a^*\mathbb{P}(X_t \in (x, x+\Delta]) \overset{t,\delta}{\backsim} f(0)U^\Delta(x),$$

where the notation $A \overset{t,\delta}{\backsim} B$ is shorthand for

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty}\frac{A}{B} = \lim_{\delta\downarrow 0}\liminf_{t\to\infty}\frac{A}{B} = 1.$$

Also, for each fixed $\delta > 0$,

$$\begin{aligned} J_2^\delta &\le \int_0^{(1-\delta)t}\int_{y=0}^{x+\Delta}\mathbb{P}(X_u \in dy)\kappa_{t-u}^\Delta((x-y)^+)du \\ &\le \frac{C(\delta)\Delta}{tc(t)}\int_0^{(1-\delta)t}\int_{y=0}^{x+\Delta}\mathbb{P}(X_u \in dy)U((x-y)^+ + \Delta)du \\ &\le \frac{C(\delta)\Delta U(x+\Delta)}{tc(t)}\int_0^{(1-\delta)t}\mathbb{P}(X_u \in (0, x+\Delta])du. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{(1-\delta)t}\mathbb{P}(X_u \in (0, x+\Delta])du &\le t_0 + \int_{t_0}^{(1-\delta)t}\mathbb{P}(X_u \in (0, x+\Delta])du \\ &\le t_0 + C\int_{t_0}^{(1-\delta)t}\frac{x+\Delta}{c(u)}du \\ &\le t_0 + \frac{C(\delta)(x+\Delta)t}{c(t)} = o(t), \end{aligned}$$

the result follows. ■

Corollary 19 *Uniformly for $0 \le x \le y = o(c(t))$ we have*

$$tc(t)\underline{n}_t((x,y]) \backsim f(0)\int_x^y U(y)dy \text{ as } t \to \infty.$$

Proof. If $y \le x = 1$ this is immediate from Proposition 18, and otherwise we split $(x, y]$ into disjoint intervals of length ≤ 1 and apply the same proposition to each interval. ■

In preparation for the next proof, we have:

Lemma 20 *The density function g of the stable meander Z_1 satisfies the identity*

$$g(x) = \int_0^1 ds\int_{y=0}^x s^{-\eta-\overline{\rho}}g(s^{-\eta}y)f_{1-s}((x-y))dy, \tag{32}$$

where f_t denotes the density function of Y_t.

Proof. We recall from [3] VIII.4 that the one dimensional law of the stable meander of length one can be written in terms of the excursion measure, $\underline{n}^Y$, of the stable process Y reflected in its past infimum by the formula

$$g(z)dz = \mathbb{P}(Z_1 \in dz) = \underline{n}^Y(\epsilon_1 \in dz | \zeta > 1), \qquad z \ge 0. \tag{33}$$

But the measure $\underline{n}^Y$ inherits the scaling property of the stable process in the form: for any $c>0$, and $s>0$,

$$\underline{n}^Y(\epsilon_s\in dy, s<\zeta)=c^{-\overline{\rho}}\underline{n}^Y(\epsilon_{s/c}\in c^{-\eta}dy, s<c\zeta), \qquad y>0, \tag{34}$$

see [3] Lemma VIII.14 or [16] for a proof of this fact. Thus

$$\begin{aligned} s^{-\eta-\overline{\rho}}g(s^{-\eta}y)dy &= s^{-\overline{\rho}}\underline{n}^Y(\epsilon_1\in s^{-\eta}dy|\zeta>1)\\ &= s^{-\overline{\rho}}\underline{n}^Y(\epsilon_s\in dz|\zeta>s)=\frac{\underline{n}^Y(\epsilon_s\in dy)}{\underline{n}^Y(\zeta>1)}, \end{aligned} \tag{35}$$

and multiplying (32) by $\underline{n}^Y(\zeta>1)dx$ we see that it reads

$$\underline{n}^Y(\epsilon_1\in dx)=\int_0^1 ds\int_{y=0}^x \underline{n}^Y(\epsilon_s\in dy)f_{1-s}((x-y))dx,$$

and this is equation (21) specialised to the stable case and $t=1$. ■

We can now prove Proposition 7, which we restate;

Proposition 21 *For all values of $\alpha\overline{\rho}$, uniformly for $x_t\ge 0$ and uniformly in Δ,*

$$\frac{c(t)\kappa_t^\Delta(x)}{\underline{n}(\zeta>t)}=\Delta(g(x_t)+o(1)) \text{ as } t\to\infty.$$

Proof. This time we write

$$\begin{aligned} t\kappa_t^\Delta(x) &= \int_0^t\int_{y=0}^{x+\Delta}\int_{z=x\vee y}^{x+\Delta}\mathbb{P}(X_u\in dy)\underline{n}_{t-u}(dz-y)du+a^*\mathbb{P}(X_t\in(x,x+\Delta])\\ &= \int_0^t\int_{y=0}^{x}\mathbb{P}(X_u\in dy)\kappa_{t-u}^\Delta(x-y)du+a^*\mathbb{P}(X_t\in(x,x+\Delta])\\ &\quad+\int_0^t\int_{y=x}^{x+\Delta}\mathbb{P}(X_u\in dy)\underline{n}_{t-u}((0,x-y+\Delta])du\\ &:= I_1+a^*\mathbb{P}(X_t\in(x,x+\Delta])+I_2. \end{aligned}$$

It is immediate from (23) that $t^{-1}\Delta^{-1}\mathbb{P}(X_t\in(x,x+\Delta])=o(\underline{n}(\zeta>t)/c(t))$. Also

$$\begin{aligned} I_2 &= \int_0^t\int_0^\Delta \mathbb{P}(X_u\in x+dz)\underline{n}_{t-u}((0,\Delta-z])du\\ &\le \int_0^t \mathbb{P}(X_u\in(x,x+\Delta])\kappa_{t-u}^\Delta(0)du\\ &= \int_0^{(1-\delta)t}+\int_{(1-\delta)t}^t \mathbb{P}(X_u\in(x,x+\Delta])\kappa_{t-u}^\Delta(0)du\\ &\le \frac{k_4(1-\delta)t\Delta U(\Delta)}{\delta tc(\delta t)}+\frac{k_0\Delta}{c((1-\delta)t)}\int_0^{\delta t}\underline{n}(\zeta>u)du\\ &\backsim \frac{k_4(1-\delta)\Delta U(\Delta_0)}{\delta^{1+\eta}c(t)}+\frac{k_0\Delta\delta^{1-\overline{\rho}}t\underline{n}(\zeta>t)}{(1-\delta)^\eta c(t)}, \end{aligned}$$

so we see that $\lim_{t\to\infty}\frac{c(t)I_2}{t\Delta\underline{n}(\zeta>t)}\le k_0\delta^\rho(1-\delta)^{-\eta}$, uniformly in x, and since δ is arbitrary, $\lim_{t\to\infty}\frac{c(t)I_2}{t\Delta\underline{n}(\zeta>t)}=0$. Next put $I_1=I_1^1+I_1^2+I_1^3$, where by

the bound (27), for large enough t

$$\begin{aligned} I_1^1 &:= \int_0^{\delta t}\int_{y=0}^{x} \mathbb{P}(X_u\in dy)\kappa^{\Delta}_{t-u}(x-y)du \\ &\le \frac{k_5\Delta \underline{n}(\zeta>(1-\delta)t)}{c((1-\delta)t)}\int_0^{\delta t}\mathbb{P}(0<X_u\le x)du \\ &\le \frac{k_5\Delta\delta t\underline{n}(\zeta>(1-\delta)t)}{c((1-\delta)t)} \backsim \frac{k_5\delta\Delta\underline{n}(\zeta>t)t}{(1-\delta)^{\overline{\rho}+\eta}c(t)}. \end{aligned}$$

Also

$$\begin{aligned} I_1^3 &:= \int_0^{\delta t}\int_{z>0}^{x} \underline{n}_u(dz)\mathbb{P}(X_{t-u}\in((x-z)^+,x-z+\Delta])du \\ &\le \frac{k_0\Delta}{c((1-\delta)t)}\int_0^{\delta t}\underline{n}(\zeta>u)du \backsim \frac{k_0\Delta\delta^{\rho}t\underline{n}(\zeta>t)}{\rho(1-\delta)^{\eta}c(t)}. \end{aligned}$$

So $\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{c(t)(I_1^1+I_1^3)}{t\Delta\underline{n}(\zeta>t)}=0$. For the other term, using Proposition 13, we have

$$\begin{aligned} I_1^2 &= \int_{\delta t}^{(1-\delta)t}\int_{z>0}^{x} \underline{n}_u(dz)\mathbb{P}(X_{t-u}\in((x-z)^+,x-z+\Delta])du \\ &= \Delta\left(\int_{\delta t}^{(1-\delta)t}\int_{z>0}^{x}\underline{n}_u(dz)f((x-z)/c(t-u))/c(t-u)\right) \\ &\quad +o\left(\Delta\int_{\delta t}^{(1-\delta)t}\int_{z>0}^{x}\frac{\underline{n}_u(dz)}{c(t-u)}du\right). \end{aligned}$$

It is easily seen that, for any fixed $\delta>0$, the error term is $o(t\Delta\underline{n}(\zeta>t)/c(t))$, and in the remaining term we write $x=c(t)x_t$, $z=c(t)y$ and $u=st$ to see that $\frac{c(t)I_2}{\Delta t\underline{n}(\zeta>t)}$ can be written as

$$\begin{aligned} &\frac{1}{t\underline{n}(\zeta>t)}\int_{\delta t}^{(1-\delta)t}\frac{c(t)du}{c(t-u)}\int_{y=0}^{x_t}\underline{n}(\epsilon_u\in c(t)dy)f(\frac{c(t)(x_t-y)}{c(t-u)})+o(1) \\ =\ &\frac{1}{\underline{n}(\zeta>t)}\int_{\delta}^{(1-\delta)}\frac{c(t)ds}{c(t(1-s))}\int_{y=0}^{x_t}\underline{n}(\epsilon_{ts}\in c(t)dy)f(\frac{c(t)(x_t-y)}{c(t(1-s)})+o(1) \\ =\ &\int_{\delta}^{(1-\delta)}\frac{\underline{n}(\zeta>ts)c(t)ds}{\underline{n}(\zeta>t)c(t(1-s))}\int_{y=0}^{x_t}\underline{n}(\epsilon_{ts}\in c(t)dy|\underline{n}(\zeta>ts))f(\frac{c(t)(x_t-y)}{c(t(1-s)})+o(1). \end{aligned}$$

It then follows from Lemma 15, the regular variation of $\underline{n}(\zeta>t)$ and the fact that f is uniformly continuous that this last expression can be written as

$$\begin{aligned} &\int_{\delta}^{(1-\delta)}\frac{ds}{s^{\overline{\rho}}(1-s)^{\eta}}\int_{y=0}^{x_t}\mathbb{P}(Z_1\in s^{-\eta}dy)f((x_t-y)(1-s)^{-\eta})+o(1) \\ =\ &\int_{\delta}^{(1-\delta)}ds\int_{y=0}^{x_t}s^{-\overline{\rho}}\mathbb{P}(Z_1\in s^{-\eta}dy)f_{1-s}((x_t-y))+o(1), \end{aligned}$$

where we have used the scaling property. It is easy to check, from the known behaviour of f and that of the density of Z_s, see [11], that the corresponding integrals over $(0,\delta)$ and $(1-\delta,1)$ are $o(1)$ as $\delta\to 0$ uniformly for $x\ge 0$, so the result follows from Lemma 20. ■

Corollary 22 *Uniformly for $x, y \geq 0$,*

$$\limsup_{t\to\infty} c(t)\underline{n}(\epsilon_t \in (x, x+y] | \zeta > t) \leq \overline{g}y,$$

where $\overline{g} = \sup_{x\geq 0} g(x) < \infty$.

Proof. If $y \leq 1$ this is immediate from Proposition 21, and otherwise we get the conclusion by writing $(x, x+y]$ as the union of disjoint intervals of length less than or equal to 1. ■

4 Proof of Theorem 1

4.1 The discontinuous case.

Here we assume $\Pi(\mathbb{R}^-) > 0$, and deal separately with the cases $\alpha\overline{\rho} < 1$ and $\alpha\overline{\rho} = 1$.

4.1.1 The case $\alpha\overline{\rho} < 1$

Write, for $y \geq 0$,

$$\begin{aligned}
\theta(t, y) &= \underline{n}(\overline{\Pi}^*(y + \epsilon_t), \zeta > t) \text{ and} \\
\chi(t, y) &= \overline{n}(\overline{\Pi}^*(y - \epsilon_t)\mathbf{1}_{\{y>\epsilon_t\}}, \zeta > t)
\end{aligned}$$

so that $\theta(t, 0) = h_0(t)$ is the density of $\underline{n}(\zeta \in dt, \epsilon(\zeta-) > 0)$. Note that, from e.g. the quintuple identity of [10] or integrating (20), we have that, for $x > 0$,

$$h_x(t) = \int_0^t \int_0^x \overline{n}_s(x - dy)\theta(t - s, y)ds + a\theta(t, x) + a^*\chi(t, y). \tag{36}$$

So as well as proving the result (4) for h_0, the following Proposition will be useful for the case $x > 0$.

Proposition 23 *Assume $\alpha\overline{\rho} < 1$. Then uniformly for $y \geq 0$,*

$$\theta(t, y) \sim \overline{\rho}t^{-1}\underline{n}(\zeta > t)\phi(y_t) \text{ as } t \to \infty, \tag{37}$$

where $y_t := y/c(t)$ and

$$\begin{aligned}
\phi(z) &= \int_0^\infty (z + w)^{-\alpha} g(w)dw \Big/ \int_0^\infty w^{-\alpha} g(w)dw \\
&= \mathbb{E}\{(z + Z_1)^{-\alpha}\}/\mathbb{E}(Z_1^{-\alpha}).
\end{aligned} \tag{38}$$

Our argument to prove Proposition 23 relies on the decomposition, for $\delta > 0$

$$\begin{aligned}
\theta(t, y) &= \int_{x>0} \underline{n}_t(dx)\overline{\Pi}^*(x + y) \\
&= \int_{\delta c(t)\geq x>0} \underline{n}_t(dx)\overline{\Pi}^*(x + y) + \int_{x>\delta c(t)} \underline{n}_t(dx)\overline{\Pi}^*(x + y) \\
&: = I_1(\delta, y) + I_2(\delta, y).
\end{aligned}$$

To deal with the first of these we need the following result.

Lemma 24 *For any Lévy process, $xU(x)\Pi^*(dx)$ is integrable at zero.*

Proof. By Vigon's identity, the tail of the Lévy measure of the down going ladder height process is given by

$$\begin{aligned}
\overline{\mu}^*(x) &= \int_0^\infty U(dy)\overline{\Pi}^*(x+y) &(39)\\
&= \int_0^\infty U(dy)\int_{x+y}^\infty \Pi^*(dz)\\
&= \int_{z=x}^\infty \Pi^*(dz)\int_{y<x-z} U(dy) = \int_{z=x}^\infty \Pi^*(dz)U(z-x)
\end{aligned}$$

So

$$\begin{aligned}
C &= \int_0^1 \overline{\mu}^*(x)dx \geq \int_{x=0}^1\int_{z=x}^1 \Pi^*(dz)U(z-x)dx\\
&= \int_{z=0}^1\int_{x=0}^z \Pi^*(dz)U(z-x)dx\\
&= \int_{z=0}^1 \Pi^*(dz)\int_{y=0}^z U(y)dy\\
&\geq \int_{z=0}^1 \Pi^*(dz)\int_{y=z/2}^z U(y)dy\\
&\geq \frac{1}{2}\int_{z=0}^1 zU(z/2)\Pi^*(dz).
\end{aligned}$$

But by Erickson's [12] bounds, $U(z/2)\geq CU(z)$, and the result follows. ■

Now we show that uniformly in $y\geq 0$

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty}\frac{tI_1(\delta,y)}{\underline{n}(\zeta>t)}=0.$$

First we note that for all $y\geq 0$, we have $I_1(\delta,y)\leq I_1(\delta,0)$. Then from Lemma 19, we can choose δ small enough and t_0 large enough such that

$$tc(t)\underline{n}_t((x,\delta c(t))\leq 2f(0)\int_x^{\delta c(t)} U(y)dy, \qquad \text{for all } 0\leq x\leq \delta c\,(t)\,.$$

And then

$$\begin{aligned}
&\int_0^{\delta c(t)} \overline{\Pi}^*(x)\underline{n}_t(dx)\\
=\;& \overline{\Pi}^*(\delta c(t))\underline{n}_t((0,\delta c(t))+\int_0^{\delta c(t)} \underline{n}_t((x,\delta c(t))\Pi^*(dx)\\
\leq\;& \frac{2f(0)}{tc(t)}\left(\overline{\Pi}^*(\delta c(t))\int_0^{\delta c(t)} U(y)dy+\int_0^{\delta c(t)}\Pi^*(dx)\int_x^{\delta c(t)} U(y)dy\right)\\
=\;& \frac{2f(0)}{tc(t)}\int_0^{\delta c(t)}\overline{\Pi}^*(x)U(x)dx \backsim \frac{C\delta c(t)\overline{\Pi}^*(\delta c(t))U(\delta c(t))}{tc(t)},
\end{aligned}$$

where we use the fact that $\overline{\Pi}^*(t)U(t)$ is rv with index $-\alpha+\alpha\rho=-\alpha\overline{\rho}>-1$. For the same reason, and using Lemma 14 we can replace the numerator by

$$\begin{aligned} C\delta^{1-\alpha+\alpha\rho}c(t)\overline{\Pi}^*(c(t))U(c(t)) &\backsim C\delta^{1-\alpha+\alpha\rho}c(t)t^{-1}t\underline{n}(\zeta>t) \\ &= C\delta^{1-\alpha+\alpha\rho}c(t)\underline{n}(\zeta>t), \end{aligned}$$

and the conclusion follows.

Next we show that for any fixed $b\geq 0$

$$\lim_{\delta\downarrow 0}\lim_{t\to\infty}\frac{tI_2(\delta,bc(t))}{\underline{n}(\zeta>t)}=\overline{\rho}\phi(b). \tag{40}$$

For this, we use Lemma 15 and write

$$\begin{aligned} \frac{tI_2(\delta,bc(t))}{\underline{n}(\zeta>t)} &= t\int_{x>\delta c(t)}\underline{n}(\epsilon_t\in dx|\zeta>t)\overline{\Pi}^*(x+bc(t)) \\ &= t\int_{y>\delta}\underline{n}(\epsilon_t\in c(t)dy|\zeta>t)\overline{\Pi}^*(c(t)(y+b)) \\ &\to k^*\int_{y>\delta}\mathbb{P}(Z_1\in dy)(y+b)^{-\alpha}dy. \end{aligned}$$

By letting $\delta\to 0$ we see that (40) holds, except that $\overline{\rho}$ is replaced by $k^*\mathbb{E}Z_1^{-\alpha}$. Taking $b=0$ this shows that $h_0(t)\backsim k^*\mathbb{E}Z_1^{-\alpha}t^{-1}\underline{n}(\zeta>t)$, and, as we show later, $\underline{n}^d(\zeta>t)=\int_t^\infty h_0(s)ds\backsim\underline{n}(\zeta>t)$. By applying this result to the case where X is an α-stable process with positivity parameter ρ we get that

$$\overline{\rho}=k^*\mathbb{E}Z_1^{-\alpha}. \tag{41}$$

We have shown that (37) holds for $y=bc(t)$. The general result then follows from the fact that $\theta(t,y)$ is monotone in y.

5 The case $\alpha\overline{\rho}=1$.

In this case the ladder height process H^* is relatively stable, i.e. there is a norming function b such that $H_t^*/b(t)\stackrel{P}{\to}1$, and this can happen in two different ways. Put $A^*(x)=\int_0^x\overline{\mu}^*(y)dy$; then either $\mathbb{E}H_1^*=d^*+A^*(\infty)<\infty$, or $A^*(\infty)=\infty$, and in the latter case $A^*\in RV(0)$. It is immediate from Vigon's identity that if we put $B(x)=\int_0^x U(y)\overline{\Pi}^*(y)dy$, then $A^*(\infty)<\infty$ iff $B(\infty)<\infty$. In our case the connection between these functions is closer than this, because:

Lemma 25 *If $\alpha\overline{\rho}=1$ and $\mathbb{E}H_1^*=\infty$ then $B(x)\backsim A^*(x)$ as $x\to\infty$.*

Proof. Integrating Vigon's identity gives

$$\begin{aligned} A^*(x) &= \int_0^x\int_0^\infty U(dz)\overline{\Pi}^*(y+z)dy \\ &= \int_0^\infty U(dz)\int_0^x\overline{\Pi}^*(y+z)dy=\int_0^\infty U(dz)\int_z^{x+z}\overline{\Pi}^*(w)dw \\ &= \int_0^\infty\overline{\Pi}^*(w)dw\int_{(w-x)^+}^w U(dz)=B(x)+E(x), \end{aligned}$$

where $E(x) = \int_x^\infty \overline{\Pi}^*(w)dw \int_{w-x}^w U(dz)$. If we put $\overline{U}(x) = \int_0^x U(y)dy$ an integration by parts gives

$$\begin{aligned} E(x) &= \int_x^\infty \Pi^*(dy) \int_x^y \{U(w) - U(w-x)\}dw \\ &= \int_x^\infty \Pi^*(dy)\{\overline{U}(y) - \overline{U}(x) - \overline{U}(y-x)\} \\ &\leq x \int_x^\infty \Pi^*(dy)U(y) = x\{\overline{\Pi}^*(x)U(x) + \int_x^\infty \overline{\Pi}^*(y)U(dy)\}. \end{aligned}$$

Since $A^{*\prime}(x) = \overline{\mu}^*(x)$ and $A^* \in RV(0)$ we know that $x\overline{\mu}^*(x) = o(A^*(x))$ as $x \to \infty$. Also

$$\begin{aligned} \overline{\mu}^*(x) &= \int_0^\infty U(dz)\overline{\Pi}^*(x+z) \geq \int_0^x U(dz)\overline{\Pi}^*(x+z) \\ &\geq U(x)\overline{\Pi}^*(2x) \geq CU(2x)\overline{\Pi}^*(2x), \end{aligned}$$

where we have used Erickson's [12] bounds for U. Thus $x\overline{\Pi}^*(x)U(x) \leq Cx\overline{\mu}^*(x/2) = o(A^*(x))$. Hence

$$x\int_x^\infty \overline{\Pi}^*(y)U(dy) = o\left(x\int_x^\infty \frac{A^*(y)U(dy)}{yU(y)}\right),$$

and we can bound the bracketed term on the RHS by

$$x \sup_{y\geq x}\left(\frac{A^*(y)y^\beta}{U(y)}\right) \bullet \int_x^\infty \frac{U(dy)}{y^{1+\beta}},$$

where we choose $\beta = \alpha\rho/2$ and recall that $U \in RV(\alpha\rho)$. From standard properties of regularly varying functions we see that this last expression is asymptotically equivalent to

$$Cx\frac{A^*(x)x^\beta}{U(x)} \bullet \frac{U(x)}{x^{1+\beta}} = CA^*(x),$$

so we can conclude that $E(x)/A^*(x) \to 0$, which gives the result. ■

This result immediately implies that the function $B(c(t))$ is monotone and slowly varying. It is therefore possible to find $\delta_t \downarrow 0$ such that $\delta_t c(t) \to \infty$ and $L(t) := B(\delta_t c(t)) \backsim B(c(t))$ is also slowly varying. Moreover, since for each fixed δ we have $t\overline{\Pi}^*(\delta c(t)) = o(t\overline{\Pi}(\delta c(t)) = o(1)$, we can also arrange that $t\overline{\Pi}^*(\delta_t c(t)) \to 0$.

Proposition 26 *Define, for $y \geq 0$, the function*

$$\psi(y,t) = \int_0^{\delta_t c(t)} U(z)\overline{\Pi}^*(z+y)dz,$$

and note that $\psi(0,t) = L(t)$. Then we have the estimate, uniform for $y \geq 0$,

$$\theta(t,y) = \frac{\overline{\rho}\psi(y,t)\underline{n}^d(\zeta > t)}{tL(t)} + o(t^{-1}\underline{n}(\zeta > t)) \ \text{as } t \to \infty.$$

In particular, $h_0(t) \backsim \overline{\rho}t^{-1}\underline{n}^d(\zeta > t)$.

Proof. Clearly, since

$$\int_{\delta_t c(t)}^{\infty} \underline{n}_t(dz)\overline{\Pi}^*(z) \leq \overline{\Pi}^*(\delta_t c(t))\underline{n}(\zeta > t) = o(t^{-1}\underline{n}(\zeta > t)),$$

we have

$$\begin{aligned}\theta(t,y) &= \int_0^{\infty} \underline{n}_t(dz)\overline{\Pi}^*(z+y) \\ &= \int_0^{\delta_t c(t)} \underline{n}_t(dz)\overline{\Pi}^*(z+y) + o(t^{-1}\underline{n}(\zeta > t)). \qquad (42)\end{aligned}$$

We can apply Proposition 6 to get

$$\begin{aligned}\int_0^{\delta_t c(t)} \underline{n}_t(dz)\overline{\Pi}^*(z+y) &= \int_0^{\delta_t c(t)} \underline{n}_t(dz) \int_{z+y}^{\infty} \Pi^*(dw) \\ &= \int_y^{\infty} \Pi^*(dw) \int_0^{(w-y)\wedge\delta_t c(t)} \underline{n}_t(dz) \\ &\backsim \frac{f(0)}{tc(t)} \int_y^{\infty} \Pi^*(dw) \int_0^{(w-y)\wedge\delta_t c(t)} U(z)dz \\ &= \frac{f(0)}{tc(t)} \int_0^{\delta_t c(t)} U(z)\overline{\Pi}^*(z+y)dz.\end{aligned}$$

In particular, we have

$$h_0(t) = \theta(t,0) = \frac{f(0)L(t)}{tc(t)} + o(t^{-1}\underline{n}(\zeta > t)),$$

and since the first term $\in RV(-(1+\eta))$ and $\eta = \overline{\rho}$ we can integrate this to give

$$\frac{f(0)L(t)}{\overline{\rho}c(t)} \backsim \underline{n}^d(\zeta > t), \qquad (43)$$

and hence $\theta(t,0) \backsim \overline{\rho}t^{-1}\underline{n}^d(\zeta > t)$. The result for $y > 0$ then follows from (42). ■

Remark 27 *The results in the following section will demonstrate that we have*
$\underline{n}^d(\zeta > t) \backsim p\underline{n}(\zeta > t)$ *and then (4) follows for the case* $\alpha\overline{\rho} = 1$.

5.1 The continuous case

It turns out that we need to establish some parts of Theorem 3 before we can conclude the proof of Theorem 1.

Theorem 28 *Suppose the drift* d^* *of* H^* *is positive. Then uniformly in* Δ *and* $x > 0$ *such that* $\frac{x}{c(t)} \to 0$,

$$\mathbb{P}_x^c(T \in (t, t+\Delta]) \backsim \frac{f(0)d^*\Delta U^*(x)}{tc(t)} \text{ as } t \to \infty, \qquad (44)$$

and uniformly in Δ *and* $x > 0$

$$\mathbb{P}^c_x(T \in (t, t+\Delta]) = \frac{d^* \Delta \overline{n}(\zeta > t)}{c(t)} \, (g^*(x_t) + o(1)) \text{ as } t \to \infty. \tag{45}$$

Proof. We use the result, from Theorem 3.1 of [13], which states that whenever $d^* > 0$ the bivariate renewal function $W^*(t, x)$ is differentiable in x for each $t > 0$, and

$$\mathbb{P}^c_x(T \le t) = d^* \frac{dW^*(t, x)}{dx}.$$

Recall also from Lemma 9 that $W^*(t, x) = a + \int_{u=0}^{t} \int_{y=0}^{x} \overline{n}_u(dy) du$, so that

$$\mathbb{P}^c_x(T \in (t, t+\Delta]) = d^* \int_t^{t+\Delta} \lim_{h \downarrow 0} \frac{\overline{n}_u((x, x+h])}{h} du.$$

However, by applying Proposition 6 to $-X$ we can approximate $\overline{n}_u((x, x+h])$ uniformly in x and h, and see that, given any $\varepsilon > 0$, for $u \in [t, t+\Delta]$, t large enough, and $x/c(t)$ small enough,

$$\frac{(1-\varepsilon) f(0) U^*(x)}{uc(u)} \le \lim_{h \downarrow 0} \frac{\overline{n}_u((x, x+h])}{h} \le \frac{(1+\varepsilon) f(0) U^*(x)}{uc(u)}$$

and then (44) is immediate. The statement (45) is proved in exactly the same way, but using the approximation from Proposition 7. ∎

For the next result, we need the following identity, in which $q_t(z)$ (respectively $q^*_t(z)$) denotes the density function $\underline{n}^Y_t(dz)/dz$ (respectively $\overline{n}^Y_t(dz)/dz$).

Lemma 29 *For any fixed* $0 < s < t$,

$$\int_0^\infty q_s(z) q^*_{t-s}(z) dz = \frac{f_t(0)}{t} = t^{-(1+\eta)} f(0). \tag{46}$$

Proof. Specialising (19) to the stable case and observing that, in the stable case both the ladder time processes have zero drift gives

$$f_t(0) = \int_0^t du \int_0^\infty q_u(z) q^*_{t-u}(z) dz.$$

Now we can deduce from Corollary 3 of [6] that $\int_0^\infty q_u(z) q^*_{t-u}(z) dz / f_t(0)$ is the conditional density function of the time at which $\sup(Y_u, 0 \le u \le t)$ occurs, given $Y_t = 0$. However it is well-known that the time at which the supremum of a stable bridge occurs has a uniform distribution, see e.g. [5] Théorème 4, and the result (46) follows. ∎

Theorem 30 *If* $d^* > 0$ *then (5) holds, viz, uniformly in* Δ,

$$\underline{n}^c(\zeta \in (t, t+\Delta]) \backsim \frac{f(0) d^* \Delta}{tc(t)} \text{ as } t \to \infty. \tag{47}$$

Proof. We will actually show that $\underline{n}^c(\zeta \in (2t, 2t+\Delta]) \backsim 2^{-(1+\eta)} f(0) d^* \Delta (tc(t))^{-1}$, which is equivalent to the stated result. Here we use a different decomposition, viz

$$\begin{aligned}
\underline{n}^c(\zeta \in (2t, 2t+\Delta]) &= \int_0^\infty \underline{n}(\epsilon_t \in dy) \mathbb{P}_y^c(T \in (t, t+\Delta]) \\
&= \sum_1^2 I_r = \sum_1^2 \int_{A_r} \underline{n}_t(dy) \mathbb{P}_y(T \in (t, t+\Delta]),
\end{aligned}$$

where $A_1 = (0, D^{-1}c(t)]$, and $A_2 = (D^{-1}c(t), \infty)$. First we have, using Corollary 19 and Theorem 28,

$$\begin{aligned}
I_1 &= \int_0^{D^{-1}c(t)} \underline{n}_t(dy) \mathbb{P}_y^c(T \in (t, t+\Delta]) \\
&\backsim \frac{d^* f(0)\Delta}{tc(t)} \int_0^{D^{-1}c(t)} \underline{n}_t(dy) U^*(y) \\
&= \frac{d^* f(0)\Delta}{tc(t)} \int_0^{D^{-1}c(t)} U^*(dz) \underline{n}(\epsilon_t \in (z, D^{-1}c(t)] \\
&\backsim \frac{d^*(f(0))^2\Delta}{(tc(t))^2} \int_0^{D^{-1}c(t)} U^*(dz) \int_z^{D^{-1}c(t)} U(y) dy.
\end{aligned}$$

Now, using Lemma 14

$$\begin{aligned}
\int_0^{D^{-1}c(t)} U^*(dz) \int_z^{D^{-1}c(t)} U(y) dy &= \int_0^{D^{-1}c(t)} U^*(z) U(z) dz \\
\leq D^{-1}c(t) U(D^{-1}c(t)) U^*(D^{-1}c(t)) &\backsim CD^{-(1+\alpha)} tc(t).
\end{aligned}$$

So we can make $\limsup_{t\to\infty} \Delta^{-1} I_1 tc(t) \leq \varepsilon$ by choice of $D = D_\varepsilon$. The result will then follow if we can show that $\lim_{D\to\infty} \lim_{t\to\infty} tc(t)(d^*\Delta)^{-1} I_2 = f(0)$. Using Theorem 28, Proposition 21, and the uniform continuity of $g(\cdot)$ and $g^*(\cdot)$, gives

$$\begin{aligned}
\frac{tc(t)}{d^*\Delta} I_2 &= \frac{tc(t)\underline{n}(\zeta > t)}{d^*\Delta} \int_{D^{-1}c(t)}^\infty \underline{n}(\epsilon_t \in dy | \zeta > t) \mathbb{P}_y^c(T \in (t, t+\Delta]) \\
&= t\overline{n}(\zeta > t)\underline{n}(\zeta > t) \int_{D^{-1}c(t)}^\infty \underline{n}(\epsilon_t \in dy | \zeta > t)(g^*(y/c(t)) + o(1)) \\
&= t\overline{n}(\zeta > t)\underline{n}(\zeta > t) \int_{D^{-1}}^\infty \underline{n}(\epsilon_t \in c(t)dz | \zeta > t)(g^*(z) + o(1)) \\
&= \frac{1}{\Gamma(\rho)\Gamma(\overline{\rho})} \int_{D^{-1}}^\infty g(z) g^*(z) dz + o(1),
\end{aligned}$$

where we have used Lemma 14. Now since

$$\begin{aligned}
g(z)dz/\Gamma(\rho) &= \underline{n}^Y(\epsilon_1 \in dz | \zeta > 1)\underline{n}^Y(\zeta > 1) = q_1(z)dz, \text{ and} \\
g^*(z)dz/\Gamma(\overline{\rho}) &= \overline{n}^Y(\epsilon_1 \in dz | \zeta > 1)\overline{n}^Y(\zeta > 1) = q_1^*(z)dz,
\end{aligned}$$

the result follows from Lemma 29. ■

Remark 31 *When $d^* > 0$ and $\mathbb{E}H_1^* < \infty$ we see from (47) and (43) that*

$$\begin{aligned} \underline{n}^c(\zeta > t) &\backsim \frac{f(0)d^*}{\overline{\rho}c(t)} \backsim q\underline{n}(\zeta > t), \\ \text{and } \underline{n}^d(\zeta > t) &\backsim \frac{f(0)A^*(\infty)}{\overline{\rho}c(t)} \backsim p\underline{n}(\zeta > t), \end{aligned}$$

where to get the second estimates we used that the first estimates imply

$$\underline{n}(\zeta > t)c(t) \to f(0)(d^* + A(\infty))/\overline{\rho}.$$

Thus we can rewrite (47) as

$$\lim_{t\to\infty} \frac{\underline{n}^c(t,\Delta]}{\overline{\rho}\Delta\underline{n}(\zeta > t)} = \frac{d^*}{(d^* + A^*(\infty))} \ ,$$

and since this also holds when $A^* = \infty$, we recover (6).

6 Proof of Theorem 3 and refinements

For the case when X is irregular upwards we need

Lemma 32 *Assume $a^* > 0$. For $\alpha\overline{\rho} \leq 1$, we have that uniformly as $x/c(t) \downarrow 0$,*

$$\chi(t,x) \begin{cases} = o(U^*(x)h_0(t)), & \text{if } \alpha\overline{\rho} < 1, \\ \sim \frac{\overline{\rho}}{d^*+A^*(\infty)} \frac{\underline{n}(\zeta>t)}{t} \int_0^x U^*(y)\overline{\Pi}^*(x-y)dy, & \text{if } \alpha\overline{\rho} = 1, \end{cases}$$

where the term $\overline{\rho}/(d^* + A^*(\infty))$ is understood as $o(1)$ when $A^*(\infty) = \infty$. Also for any $D > 0$, uniformly in $D^{-1}c(t) < x < Dc(t)$,

$$t\chi(t,x) = o(1).$$

Proof. First observe that the fact that $a^* > 0$ implies that X is irregular upwards and, by Bertoin's test, see e.g. page 64 in [9], necessarily X has bounded variation. A consequence of the bounded variation of X is that

$$\int_{\mathbb{R}\setminus\{0\}} 1 \wedge |w|\Pi(dw) < \infty, \quad y\overline{\Pi}^*(y) = o(1), \quad \text{as } y \to 0.$$

Making an integration by parts it is easily seen that

$$\chi(t,x) = \int_0^\infty \Pi^*(dw)\overline{n}_t((x-w)^+ < \epsilon_t < x).$$

Assume that $x_t \to 0$ as $t \to \infty$. By the usual approximation method using Lemma 18 we have that uniformly in $x_t \to 0$ as $t \to \infty$,

$$\begin{aligned} \chi(t,x) &\sim \frac{f(0)}{tc(t)} \left(\int_0^x \Pi^*(dw) \int_{(x-w)^+}^x U^*(z)dz \right) \\ &= \frac{f(0)}{tc(t)} \int_0^x U^*(z)\overline{\Pi}^*(x-z)dz. \end{aligned}$$

When $\alpha\overline{\rho} = 1$, Lemma 14 and the elementary renewal theorem imply that

$$\frac{1}{c(t)\underline{n}(\zeta > t)} \sim \frac{U^*(c(t))}{c(t)k_1} \xrightarrow[t\to\infty]{} \frac{1}{k_1\mathbb{E}(H_1^*)},$$

where the above is understood as zero when $\mathbb{E}(H_1^*) = \infty$. Remark 31 implies that when $\alpha\overline{\rho} = 1$, then the above limit equals $\overline{\rho}/f(0)\mathbb{E}(H_1^*)$. So the result follows by equating the constants.

In the case where $\alpha\overline{\rho} < 1$, we can chose t large enough such that $x < c(t)$ and thus we have that

$$\begin{aligned}
\frac{t}{\underline{n}(\zeta > t)U^*(x)}\chi(t,x) &\sim \frac{f(0)}{c(t)\underline{n}(\zeta > t)}\frac{1}{U^*(x)}\int_0^x U^*(z)\overline{\Pi}^*(x-z)dz \\
&\leq \frac{f(0)\int_0^x \overline{\Pi}^*(z)dz}{c(t)\underline{n}(\zeta > t)} \\
&\sim C\frac{U^*(c(t))}{c(t)}\int_0^x \overline{\Pi}^*(z)dz \\
&\leq C\frac{\int_0^{c(t)} \overline{\Pi}^*(z)dz}{\int_0^{c(t)} \overline{\mu}^*(y)dy} \\
&= o(1),
\end{aligned}$$

in the third line we used Lemma 14, in the fourth line we used Proposition III.1 in [3], in the fifth line we used that $\int_0^{c(t)} \overline{\Pi}^*(z)dz \in RV((1-\alpha)^+/\alpha)$, $\int_0^{c(t)} \overline{\mu}^*(y)dy \in RV((1-\alpha\overline{\rho})/\alpha)$ and that $(1-\alpha)^+ < (1-\alpha\overline{\rho})$.

We now deal with the case $D^{-1}c(t) < x < Dc(t)$. As before by the usual approximation method using Lemma 21 we have that

$$\begin{aligned}
\chi(t,x) &\sim \frac{\overline{n}(\zeta > t)}{c(t)}\int_0^x dw\overline{\Pi}^*(w)\left(g^*\left(\frac{(x-w)^+}{c(t)}\right) + o(1)\right) \\
&\leq C\frac{\overline{n}(\zeta > t)}{c(t)}\int_0^{Dc(t)} dw\overline{\Pi}^*(w).
\end{aligned}$$

Observe that, by Karamata's Theorem, in all cases $\int_0^{Dc(t)} dw\overline{\Pi}^*(w) = o(c(t))$, so the result follows. ■

6.1 The small deviation case

Theorem 33 *If X is asymptotically stable with $\alpha\overline{\rho} < 1$, then uniformly in $x > 0$ such that $x_t := x/c(t) \to 0$,*

$$h_x(t) \backsim U^*(x)h_0(t) \backsim p\overline{\rho}U^*(x)\underline{n}(\zeta > t)/t \text{ as } t \to \infty.$$

Remark 34 *Since $\mathbb{E}H_1^* = \infty$ we know, by Theorem 28 and remark 31, that $\underline{n}^c(t,\Delta] = o(\underline{n}^d(t,\Delta])$ and $\mathbb{P}_x^c(T_0 \in (t,t+\Delta]) = o(U^*(x)\underline{n}(\zeta > t)/t)$, and since $p = 1$ this will give the result of Theorem 3 when $\alpha\overline{\rho} < 1$, and also the analogue of (4).*

Proof. Recalling equation (36) and Lemma 32 we can write $h_x(t) = I_1 + I_2 + a\theta(t,x) + a^* o(U^*(x)h_0(t))$ where

$$\begin{aligned}
I_1 + a\theta(t,x) &= \int_0^{\delta t} ds \int_0^x \overline{n}_s(x-dy)\theta(t-s,y) + a\theta(t,x) \\
&= \int_{[0,\delta t)} \int_{[0,x]} W^*(ds, x-dy)\theta(t-s,y) \\
&\backsim \overline{\rho} \int_{[0,\delta t)} \int_{[0,x]} W^*(ds,x-dy)(t-s)^{-1}\underline{n}(\zeta > t-s)\phi(y/c(t-s)),
\end{aligned}$$

uniformly in x, by Proposition 23. Since $\phi \le 1$ and it is a non-increasing function we can bound the latter from above by

$$\frac{\overline{\rho}\underline{n}(\zeta > t(1-\delta))}{t(1-\delta)} \int_{[0,\delta t)} \int_{[0,x]} W^*(ds,x-dy) \le \frac{\overline{\rho}\underline{n}(\zeta > t(1-\delta))U^*(x)}{t(1-\delta)},$$

and below by

$$\begin{aligned}
&\frac{\overline{\rho}\underline{n}(\zeta > t)\phi(x/c(t))}{t} \int_{[0,\delta t)} \int_{[0,x]} W^*(ds,x-dy) \\
\ge\ & \frac{(1-\varepsilon)\overline{\rho}\underline{n}(\zeta > t)}{t}\left(U^*(x) - \int_{\delta t}^{\infty}\int_{[0,x]} W^*(ds,x-dy)\right)
\end{aligned}$$

for arbitrary $\varepsilon > 0$ and all sufficiently large t. Also, using the result corresponding to Proposition 6 for $-X$

$$\begin{aligned}
\int_{\delta t}^{\infty}\int_0^x W^*(ds,x-dy) &= \int_{\delta t}^{\infty} ds \int_0^x \overline{n}_s(dy) \\
&\le C\int_{\delta t}^{\infty} ds \int_0^x U^*(y)dy/sc(s) \\
&\le CxU^*(x)/c(\delta t) = o(U^*(x)),
\end{aligned}$$

and we conclude that

$$I_1 + a\theta(t,x) \overset{t,\delta}{\backsim} h_0(t)U^*(x).$$

Also, we can write $\theta(t,y) = \int_y^\infty \nu(t,dw)$ where $\nu(t,dw) = \int_0^\infty \underline{n}_t(dz)\Pi^*(dw+z)$. This allows us to integrate $\int_0^x \overline{n}_{t-s}(dy)\theta(s,x-y)$ by parts and apply the result for $-X$ corresponding to Corollary 19, to get

$$\begin{aligned}
I_2 &= \int_0^{(1-\delta)t} ds \int_0^x \overline{n}_{t-s}(dy)\theta(s,x-y) \\
&\le \frac{C}{tc(t)}\int_0^{(1-\delta)t} ds \int_0^x U^*(y)\theta(s,x-y)dy \\
&\le \frac{C}{tc(t)}\int_0^x U^*(y)\underline{n}\{O > x-y\}dy \\
&\le \frac{CU^*(x)A^*(x)}{tc(t)}
\end{aligned}$$

where we recall that $A^*(x)=\int_0^x \overline{\mu}^*(y)dy$, $\overline{\mu}^*(y)=\underline{n}(O>y)$ is the tail of the Lévy measure of the decreasing ladder-height process, and $U^*(x)\backsim x/A^*(x)$ as $x\to\infty$. Since $A^*\in RV(1-\alpha\overline{\rho})$ we have

$$\begin{aligned}
A^*(x)/c(t)\underline{n}(\zeta>t) &= o(A^*(c(t))/c(t)\underline{n}(\zeta>t) \\
&= o(1/U^*(c(t))\underline{n}(\zeta>t)),
\end{aligned}$$

and the result follows from Lemma 14. ■

Theorem 35 *If X is asymptotically stable with $\alpha\overline{\rho}=1$, the conclusion of Theorem 33 holds.*

Proof. This time we write $h_x(t)=I_1+I_2+I_3+a\theta(t,x)+a^*\chi(t,x)$ where

$$\begin{aligned}
I_1+a\theta(t,x) &= \int_0^{\delta t} ds\int_0^x \overline{n}_s(x-dy)\theta(t-s,y)+a\theta(t,x) \\
&= \int_0^{\delta t}\int_{(0,x]} W^*(ds,x-dy)\theta(t-s,y).
\end{aligned}$$

Since $\int_0^{\delta t}\int_{(0,x]} W^*(ds,x-dy)\le U^*(x)$ we see from Proposition 26 that, writing $\Delta_t=\delta_t c(t)$ and introducing the monotone decreasing function $\gamma(t)=\overline{\rho}\underline{n}(\zeta>t)/(tL(t))$,

$$I_1=\int_0^{\delta t}\int_{(0,x]}\int_{z=0}^{\Delta_t} W^*(ds,x-dy)\gamma(t-s)U(z)\overline{\Pi}^*(z+y)dz+o(U^*(x)\underline{n}(\zeta>t)/t).$$

The integral here is bounded above by $\gamma((1-\delta)t)J(t,x)$ and below by $\gamma(t)(J(t,x)-e(t,x))$, where

$$\begin{aligned}
J(t,x) &= \int_{0<y\le x}\int_{z=0}^{\Delta_t} U^*(x-dy)U(z)\overline{\Pi}^*(z+y)dz, \\
e(t,x) &= \int_{\delta t}^{\infty}\int_{0<y\le x}\int_{z=0}^{\Delta_t} \overline{n}_s(x-dy)U(z)\overline{\Pi}^*(z+y)dzds.
\end{aligned}$$

Note that

$$\begin{aligned}
e(t,x) &\le \int_{\delta t}^{\infty}\int_{0<y\le x}\int_{z=0}^{\Delta_t} \overline{n}_s(x-dy)U(z)\overline{\Pi}^*(z)dzds \\
&= L(t)\int_{\delta t}^{\infty}\overline{n}_s((0,x])ds \backsim L(t)f(0)\int_0^x U^*(y)dy\int_{\delta t}^{\infty}\frac{ds}{sc(s)} \\
&\backsim \frac{\alpha L(t)f(0)}{\delta^{\eta}c(t)}\int_0^x U^*(y)dy\le\frac{\alpha f(0)}{\delta^{\eta}}\frac{xU^*(x)L(t)}{c(t)}=o(U^*(x)\underline{n}(\zeta>t)/t),
\end{aligned}$$

where we have used Corollary 19 in the second line. Also

$$\begin{aligned}
& J(t,x) \qquad (48)\\
= & \int_{z=0}^{\Delta_t} U(z)\int_0^x U^*(x-dy)\overline{\Pi}^*(z+y)dz\\
= & \int_{z=0}^{\Delta_t} U(z)dz\left(U^*(x)\overline{\Pi}^*(z)-\int_0^x U^*(x-y)\Pi^*(z+dy)\right)\\
= & U^*(x)L(t)-\int_{z=0}^{\Delta_t} U(z)dz\int_z^{z+x} U^*(x+z-w)\Pi^*(dw)\\
= & U^*(x)L(t)-\int_{w=0}^{\Delta_t+x}\Pi^*(dw)\int_{(w-x)^+}^{w} U^*(x+z-w)U(z)dz\\
= & U^*(x)L(t)-\int_{w=0}^{\Delta_t+x}\Pi^*(dw)\int_{(w-\Delta_t)^+}^{x\wedge w} U^*(x-y)U(w-y)dy. \qquad (49)
\end{aligned}$$

Also, using Proposition 6 and the usual approximation argument, we see that

$$\begin{aligned}
I_3 &= \int_0^{\delta t} ds\int_0^x \theta(s,y)\overline{n}_{t-s}(x-dy)\\
&\backsim \int_0^{\delta t}\int_0^x \frac{f(0)\theta(s,y)}{(t-s)c(t-s)}U^*(x-y)dyds\\
&\le \frac{f(0)}{(1-\delta)tc((1-\delta)t)}\int_0^x\int_0^\infty \theta(s,y)U^*(x-y)dyds.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^\infty \theta(s,y)ds &= \int_0^\infty\int_0^\infty \underline{n}_s(dz)\overline{\Pi}^*(y+z)ds\\
&= \int_0^\infty U(dz)\overline{\Pi}^*(y+z)-a^*\overline{\Pi}^*(y)=\overline{\mu}^*(y)-a^*\overline{\Pi}^*(y),
\end{aligned}$$

we get that the double integral above equals

$$\int_0^x \overline{\mu}^*(y)U^*(x-y)dy-a^*\int_0^x dy\overline{\Pi}^*(y)U^*(x-y).$$

Noting that $\int_{\delta t}^\infty \theta(s,y)ds\le \underline{n}^d(\zeta>\delta t)$ and so

$$\begin{aligned}
& \frac{1}{tc(t)}\int_0^x U^*(x-y)dy\int_{\delta t}^\infty \theta(s,y)ds \le \frac{\underline{n}(\zeta>\delta t)}{tc(t)}\int_0^x U^*(x-y)dy\\
\le & \frac{xU^*(x)\underline{n}(\zeta>\delta t)}{tc(t)}=o(t^{-1}U^*(x)\underline{n}(\zeta>t)),
\end{aligned}$$

and recalling that $f(0)/tc(t)\backsim \overline{\rho}\underline{n}^d(\zeta>t)/tL(t)=p\gamma(t)$, we see that there is a corresponding lower bound and hence, from Lemma 32,

$$\lim_{\delta\to 0,t\to\infty}\frac{I_3+a^*\chi(t,x)}{\gamma(t)K(x)}=p,\ \text{where } K(x)=\int_0^x \overline{\mu}^*(y)U^*(x-y)dy. \qquad (50)$$

On the other hand, using Vigon's expression for $\overline{\mu}^*$ we see that

$$\begin{aligned} K(x) &= \int_0^x \int_0^\infty \Pi^*(y+dv)U(v)U^*(x-y)dy \\ &= \int_0^\infty \Pi^*(du)\int_0^{x\wedge u} U(u-y)U^*(x-y)dy, \end{aligned}$$

and hence

$$\begin{aligned} J(t,x)+K(x)-U^*(x)L(t) &= \int_{x+\Delta_t}^\infty \Pi^*(du)\int_0^{x\wedge u} U(u-y)U^*(x-y)dy \\ &\leq U^*(x)\int_{x+\Delta_t}^\infty \Pi^*(du)\int_0^{(x+\Delta_t)} U(u-y)dy \\ &= U^*(x)E(x+\Delta_t) = o(U^*(x)A^*(x+\Delta_t)), \end{aligned}$$

by Lemma 25. But for large t we have $\Delta_t \leq x+\Delta_t \leq c(t)$, so $A^*(x+\Delta_t) \backsim L(t)$. Then it follows from (49) and (50) that, uniformly in x,

$$\lim_{\delta\to 0, t\to\infty} \frac{t(I_1+I_3+a^*\chi(t,x))}{\overline{\rho}U^*(x)\underline{n}(\zeta>t)} = p.$$

It is also straight forward to check that, for any fixed $\delta \in (0,1/2)$, $I_2 = o(t^{-1}U^*(x)\underline{n}(\zeta>t))$, and the result follows. ■

6.2 Normal deviations

Again we start with a preparatory result.

Lemma 36 *If $\alpha\overline{\rho}<1$, the identity*

$$\widetilde{h}_x(1) = \frac{\overline{\rho}}{\Gamma(\overline{\rho})\Gamma(\rho)}\int_0^1 ds\int_0^x dy\phi\left((x-y)(1-s)^{-\eta})\right)(1-s)^{-\overline{\rho}-1}g^*\left(ys^{-\eta}\right)s^{-\rho-\eta}, \tag{51}$$

holds for $x>0$, where ϕ is defined in Proposition 23 and $\tilde{h}_x$ is the downwards first passage density for Y starting from $x>0$.

Proof. Recall that $\phi(z) = \mathbb{E}(z+Z_1)^{-\alpha}/\mathbb{E}Z_1^{-\alpha} = k^*\mathbb{E}(z+Z_1)^{-\alpha}/\overline{\rho}$, where we have used (41). Also the left-hand tail of the Lévy measure of Y is $k^*x^{-\alpha}$, so if we write the equation (36) for Y with $t=1$ we have

$$\begin{aligned} \widetilde{h}_x(1) &= k^*\int_0^1 ds\int_0^x \overline{n}_s^Y(dy)\underline{n}^Y((x-y+\epsilon_{1-s})^{-\alpha}, \zeta>1-s) \\ &= k^*\int_0^1 ds\int_0^x\int_0^\infty q_s^*(y)(x-y+z)^{-\alpha}q_{1-s}(z)dydz. \end{aligned}$$

Using (35) and its analogue for q^*, and recalling that $\underline{n}^Y(\zeta > 1)\overline{n}^Y(\zeta > 1) = (\Gamma(\overline{\rho})\Gamma(\rho))^{-1}$ the RHS becomes

$$\begin{aligned}
& \frac{k^*}{\Gamma(\overline{\rho})\Gamma(\rho)} \int_0^1 ds \int_0^x \int_0^\infty (x-y+z)^{-\alpha} s^{-\eta-\rho} g^*(ys^{-\eta})(1-s)^{-\eta-\overline{\rho}} g(z(1-s)^{-\eta}) dydz \\
= & \frac{k^*}{\Gamma(\overline{\rho})\Gamma(\rho)} \int_0^1 ds \int_0^x \int_0^\infty (x-y+w(1-s)^{\eta})^{-\alpha} s^{-\eta-\rho} g^*(ys^{-\eta})(1-s)^{-\overline{\rho}} g(w) dydw \\
= & \frac{k^*}{\Gamma(\overline{\rho})\Gamma(\rho)} \int_0^1 ds \int_0^x \int_0^\infty ((x-y)(1-s)^{-\eta}+w)^{-\alpha} s^{-\eta-\rho} g^*(ys^{-\eta})(1-s)^{-1-\overline{\rho}} g(w) dydw \\
= & \frac{\overline{\rho}}{\Gamma(\overline{\rho})\Gamma(\rho)} \int_0^1 ds \int_0^x \phi((x-y)(1-s)^{-\eta}) s^{-\eta-\rho} g^*(ys^{-\eta})(1-s)^{-1-\overline{\rho}} dy,
\end{aligned}$$

and the result follows. ■

Theorem 37 *Assume $\alpha\overline{\rho} < 1$. Then uniformly for $x_t \in [D^{-1}, D]$,*

$$th_x(t) = p\tilde{h}_{x_t}(1) + o(1) \text{ as } t \to \infty.$$

Proof. Recall again that $p = 1$ in this situation. We use the same decomposition as in the proof of Theorem 33. Then

$$\begin{aligned}
I_1 + a\theta(t,x) &= \int_0^{\delta t} ds \int_0^x \overline{n}_s(x-dy)\theta(t-s,y) + a\theta(t,x) \\
&= \int_0^{\delta t} \int_{(0,x]} W^*(ds, x-dy)\theta(t-s,y) \\
&\le \int_0^{\delta t} W^*(ds,[0,\infty))\theta(t-s,0) \\
&\le Ch_0((1-\delta)t)V^*(\delta t) \backsim C\delta^{\overline{\rho}} t^{-1} \underline{n}(\zeta > t) V^*(t) \\
&\backsim C\delta^{\overline{\rho}} t^{-1}.
\end{aligned}$$

(Recall that V^* is the potential function in the decreasing ladder time process.) Next, take $0 < \gamma < D^{-1}$,and write $I_3 = I_3^1 + I_3^2$, where

$$\begin{aligned}
I_3^1 &= \int_{(1-\delta)t}^t ds \int_0^{\gamma c(t)} \overline{n}_s(x-dy)\theta(t-s,y) \\
&= \int_{(1-\delta)t}^t ds \int_0^{\gamma c(t)} \overline{n}_s(x-dy) \int_0^\infty \underline{n}_{t-s}(du)\overline{\overline{\Pi}}^*(y+u) \\
&= \int_{(1-\delta)t}^t ds \int_0^\infty \underline{n}_{t-s}(du) \int_0^{\gamma c(t)} \overline{n}_s(x-dy) \int_{y+u}^\infty \Pi^*(dw) \\
&= \int_{(1-\delta)t}^t ds \int_0^\infty \underline{n}_{t-s}(du) \int_u^\infty \Pi^*(dw) \int_0^{\gamma c(t)\wedge(w-u)} \overline{n}_s(x-dy).
\end{aligned}$$

From Corollary 22 we see that for all $\gamma > 0$ and all $s \ge (1-\delta)t$ and all sufficiently large t,

$$\int_0^{\gamma c(t)\wedge(w-u)} \overline{n}_s(x-dy) \le \frac{C\overline{n}(\zeta > t)\int_0^{\gamma c(t)\wedge(w-u)} dy}{c(t)},$$

and hence

$$\int_u^\infty \Pi^*(dw)\int_0^{\gamma c(t)\wedge(w-u)} \overline{n}_s(x-dy) \leq \frac{C\overline{n}(\zeta>t)\int_u^\infty \Pi^*(dw)\int_0^{\gamma c(t)\wedge(w-u)} dy}{c(t)}$$
$$= \frac{C\overline{n}(\zeta>t)\int_0^{\gamma c(t)} dy\overline{\Pi}^*(u+y)}{c(t)}.$$

Thus

$$\begin{aligned}
c(t)I_3^1 &\leq C\overline{n}(\zeta>t)\int_0^{\delta t} ds\int_0^\infty \underline{n}_s(du)\int_0^{\gamma c(t)} \overline{\Pi}^*(u+y)dy \\
&= C\overline{n}(\zeta>t)\int_0^{\delta t}\int_0^\infty W(ds,du)\int_0^{\gamma c(t)} \overline{\Pi}^*(u+y)dy \\
&\leq C\overline{n}(\zeta>t)\int_0^\infty U(du)\int_0^{\gamma c(t)} \overline{\Pi}^*(u+y)dy \\
&= C\overline{n}(\zeta>t)\int_{z=0}^{\gamma c(t)} \overline{\mu}^*(z)dz \backsim C\overline{n}(\zeta>t)\gamma c(t)\overline{\mu}^*(\gamma c(t)) \\
&\backsim \frac{C\gamma\overline{n}(\zeta>t)c(t)}{U^*(\gamma c(t))} \backsim \frac{C\gamma^{1-\alpha\overline{\rho}}\overline{n}(\zeta>t)c(t)}{U^*(c(t))} \\
&\backsim C\gamma^{1-\alpha\overline{\rho}}\overline{n}(\zeta>t)c(t)\underline{n}(\zeta>t) \backsim C\gamma^{1-\alpha\overline{\rho}}c(t)t^{-1}.
\end{aligned}$$

Thus $\lim_{\gamma\to 0}\limsup tI_3^1 = 0$. Also

$$\begin{aligned}
I_3^2 &= \int_{(1-\delta)t}^t ds\int_{\gamma c(t)}^x \overline{n}_s(x-dy)\underline{n}(\overline{\Pi}^*(y+\epsilon_{t-s}), t-s<\zeta) \\
&\leq \overline{\Pi}^*(\gamma c(t))\int_{(1-\delta)t}^t ds\int_{\gamma c(t)}^x \overline{n}_s(x-dy)\underline{n}(\zeta>t-s) \\
&\leq \overline{\Pi}^*(\gamma c(t))\mathbb{P}(G_t \geq (1-\delta)t),
\end{aligned}$$

where G_t, the time of the last zero of $X-I$ before t, has the property that $t^{-1}G_t$ has a limiting arc-sine distribution of index $\overline{\rho}$. (See Theorem 14, p 169 of [3].) It follows that for each fixed $\gamma>0$, we have $\lim_{\delta\to 0}\limsup_{t\to\infty} tI_3^2 = 0$, and hence $\lim_{\delta\to 0}\limsup_{t\to\infty} t(I_1+I_3) = 0$, uniformly in x. The term $a^*\chi(t,x)$ is $o(t^{-1})$ by Lemma 32. Using the bounds

$$\begin{aligned}
tI_2 &\geq t\int_{\delta t}^{(1-\delta)t} ds\sum_0^{[x]} \overline{n}_s((r,r+1])\theta(t-s,(x-r)) \\
tI_2 &\leq t\int_{\delta t}^{(1-\delta)t} ds\sum_0^{[x]} \overline{n}_s((r,r+1])\theta(t-s,(x-r-1)^+)
\end{aligned}$$

and Propositions 23 and 21, for any $\delta>0$, we can estimate tI_2 by

$$k_3\overline{\rho}\int_{\delta t}^{(1-\delta)t} ds\sum_0^{[x]} \frac{g^*(r/c(s)\overline{n}(\zeta>s)\underline{n}(\zeta>t-s)\phi((x-r-1)^+/c\,(t-s))}{c(s)\overline{n}(\zeta>t)\underline{n}(\zeta>t)(t-s)}(1+o(1)),$$

where the error term is uniform in x. Putting $r = c(t)z$ and $s = tu$ we get the uniform estimate

$$k_3\overline{\rho}\int_\delta^{1-\delta}\int_0^{x_t} g^*(zu^{-\eta})u^{-(\eta+\rho)}\phi((x_t - z)(1-u)^{-\eta})(1-u)^{-1-\overline{\rho}}dudz + o(1)$$
$$:= I(\delta, x_t) + o(1).$$

Next, we show that, as $\delta \to 0$, $I(\delta, w) = I(0, w) + o(1)$, uniformly in w. First, since ϕ is bounded, for small δ

$$\begin{aligned} & \int_0^\delta\int_0^w g^*(zu^{-\eta})u^{-(\eta+\rho)}\phi((w-z)(1-u)^{-\eta})(1-u)^{-(2-\rho)}dudz \\ \leq \quad & C\int_0^\delta\int_0^w g^*(zu^{-\eta})u^{-(\eta+\rho)}dudz = c\int_0^\delta\int_0^{wu^{-\eta}} g^*(y)u^{-\rho}dudy \\ \leq \quad & C\int_0^\delta\int_0^\infty g^*(y)u^{-\rho}dudy \to 0 \text{ as } \delta \to 0. \end{aligned}$$

Also g^* is bounded, so the same argument shows that the contribution from $(1-\delta, 1)$ is bounded above by $C\int_0^\delta\int_0^{Du^{-\eta}}\phi(z)u^{\eta+\rho-2}dudz$. By considering separately the cases $\alpha < 1, \alpha = 1$, and $\alpha > 1$, it is easy to check that this is also finite and $\to 0$ as $\delta \to 0$, and then the result follows from Lemma 36. ■

Theorem 38 *If X is asymptotically stable with $\alpha\overline{\rho} = 1$, then uniformly for $x_t \in [D^{-1}, D]$,*

$$h_x(t) = \frac{\overline{n}(\zeta > t)L(t)}{c(t)}(g^*(x_t) + o(1)) \text{ as } t \to \infty.$$

Proof. Notice that, by (43) and Remark 31

$$\frac{t\overline{n}(\zeta > t)L(t)}{c(t)} \backsim \overline{\rho}f(0)t\overline{n}(\zeta > t)\underline{n}^d(\zeta > t) \to pk_3\overline{\rho}f(0) := k_6, \tag{52}$$

so we will prove that $th_x(t) = k_6 g^*(x_t) + o(1)$. This time we write

$$\begin{aligned} h_x(t) &= \int_0^t\int_0^x \overline{n}_s(x-dy)\theta(t-s,y) + a\theta(t,x) + a^*\chi(t,x) \\ &= \sum_1^4 J_r + a\theta(t,x) + a^*\chi(t,x), \end{aligned}$$

where

$$\begin{aligned} J_1 &= \int_0^{\delta t}\int_{\Delta_t}^x \overline{n}_s(x-dy)\theta(t-s,y) \\ &\leq \int_0^{\delta t}\int_{\Delta_t}^x \overline{n}_s(x-dy)\theta(t-s,0) \\ &\leq \frac{C\underline{n}(\zeta > (1-\delta t))}{(1-\delta)t}\int_0^{\delta t}\int_{\Delta_t}^x W^*(ds, x-dy) \\ &\leq \frac{C\underline{n}(\zeta > (1-\delta t))U^*(\Delta_t)}{(1-\delta)t}, \end{aligned}$$

where $\Delta_t = c(t)\delta_t$ and δ_t has been defined before Proposition 26. Since $U^* \in RV(1)$ we see that $U^*(\Delta_t) = o(U^*(c(t)) = o((\underline{n}(\zeta > t))^{-1})$, so $\lim_{t\to\infty} tJ_1 = 0$ for any fixed $\delta > 0$. Next, we can use Proposition 7 and the usual approximation procedure to see that

$$\begin{aligned}
tJ_2 &= \int_{\delta t}^{t}\int_{\Delta_t}^{x} \overline{n}_s(x-dy)\theta(t-s,y)ds \\
&\backsim t\int_0^{(1-\delta)t}\int_{\Delta_t}^{x} \frac{\overline{n}(\zeta > t-s)g^*((x-y)/c(t-s))\theta(s,y)}{c(t-s)}dyds \\
&\leq \frac{Ct\overline{n}(\zeta > (1-\delta)t)}{\delta c(\delta t)}\int_0^{\infty}\int_{\Delta_t}^{x}\theta(s,y)dyds \\
&\leq \frac{Ct\overline{n}(\zeta > (1-\delta)t)}{c(\delta t)}\int_{\Delta_t}^{Dc(t)}\overline{n}(O > y)dy \\
&\backsim \frac{C(A^*(Dc(t)) - A^*(\Delta_t))}{c(t)\underline{n}(\zeta > t)} \\
&\backsim \frac{CU^*(c(t))(A^*(Dc(t)) - A^*(\Delta_t))}{\delta^{\rho}(1-\delta)^{\eta}c(t)} \\
&\backsim \frac{C(A^*(Dc(t)) - A^*(\Delta_t))}{\delta^{\rho}(1-\delta)^{\eta}A^*(c(t))} \to 0,
\end{aligned}$$

again for any fixed $\delta > 0$. (In the final step we have used (43) and Lemma 25.) Also

$$\begin{aligned}
tJ_3 &= t\int_0^{(1-\delta)t}\int_0^{\Delta_t}\overline{n}_s(x-dy)\theta(t-s,y)ds \\
&\leq t\int_0^{(1-\delta)t}\int_0^{\Delta_t}W^*(ds,x-dy)\theta(t-s,0) \\
&\leq Cth_0(\delta t)\int_0^{\infty}\int_0^{\Delta_t}W^*(ds,x-dy) \\
&\backsim C\delta^{-(1+\overline{\rho})}\underline{n}(\zeta > t)(U^*(x) - U^*(x-\Delta_t)) \\
&\leq C\delta^{-(1+\overline{\rho})}\underline{n}(\zeta > t)U^*(\Delta_t) \\
&\backsim C\delta^{-(1+\overline{\rho})}\frac{C\delta^{-(1+\overline{\rho})}U^*(\delta_t c(t))}{U^*(c(t))} \backsim C\delta^{-(1+\overline{\rho})}\delta_t \to 0.
\end{aligned}$$

Finally, arguing as for J_2 gives

$$\begin{aligned}
tJ_4 &= t\int_{(1-\delta)t}^{t}\int_0^{\Delta_t}\overline{n}_s(x-dy)\theta(t-s,y)ds \\
&\backsim t\int_0^{\delta t}\int_0^{\Delta_t}\frac{\overline{n}(\zeta > t-s)g^*((x-y)/c(t-s))\theta(s,y)}{c(t-s)}dyds \\
&\backsim tg^*(x_t)\int_0^{\delta t}\int_0^{\Delta_t}\frac{\overline{n}(\zeta > t-s)\theta(s,y)}{c(t-s)}dyds.
\end{aligned}$$

An upper bound for the integral here is

$$\begin{aligned}
&\frac{\overline{n}(\zeta > (1-\delta)t)}{c((1-\delta)t)}\int_0^{\infty}\int_0^{\Delta_t}\theta(s,y)dyds \\
=\ &\frac{\overline{n}(\zeta > (1-\delta)t)}{c((1-\delta)t)}\int_0^{\Delta_t}\overline{\mu}^*(y)dy \backsim \frac{\overline{n}(\zeta > t)L(t)}{(1-\delta)c(t)}.
\end{aligned}$$

An asymptotic lower bound is

$$\frac{\overline{n}(\zeta > t)}{c(t)} \left(\int_0^{\Delta_t} \overline{\mu}^*(y) dy - \int_{\delta t}^{\infty} \int_0^{\Delta_t} \theta(s,y) dy ds \right),$$

and since

$$\begin{aligned}
\int_{\delta t}^{\infty} \int_0^{\Delta_t} \theta(s,y) dy ds &\leq \int_{\delta t}^{\infty} \int_0^{\Delta_t} \theta(s,0) dy ds \\
&= \Delta_t \int_{\delta t}^{\infty} h_0(s) ds \\
&= \delta_t c(t) \underline{n}^d(\zeta > \delta t) \\
&\backsim (\overline{\rho} f(0))^{-1} \delta^{-\overline{\rho}} \delta_t L(t),
\end{aligned}$$

it follows that $\lim_{\delta \to 0, t \to \infty} \frac{tJ_4}{g^*(x_t)} = k_6$, uniformly for $x_t \in [D^{-1}, D]$. The result follows, using Lemma 32 to estimate $\chi(t,x)$ ■

Corollary 39 *Whenever* $\Pi((-\infty, 0)) > 0$ *we have*

$$th_x(t) = p\tilde{h}_{x_t}(1) + o(1) \text{ as } t \to \infty, \tag{53}$$

and in all cases (8) of Theorem 3 holds.

Proof. We have proved (53) for the case $\alpha\overline{\rho} < 1$ in Theorem 37, and in Proposition 14 of [8] it was shown that when $\alpha\overline{\rho} = 1$ there is a constant k_7 such that $g^*(x) = k_7\tilde{h}_x(1)$, so in this case we need to check that $k_6k_7 = p$. But we have, from Theorems 28 and 38,

$$\begin{aligned}
t\mathbb{P}_x^c(T &\in (t, t+\Delta]) \backsim \frac{d^*\Delta k_6 k_7 \tilde{h}_x(1)}{L(t)}, \\
t\mathbb{P}_x^d(T &\in (t, t+\Delta]) \backsim \Delta k_6 k_7 \tilde{h}_x(1).
\end{aligned}$$

If $p = 1$, i.e. $d^* = 0$ or $d^* > 0$ and $L(\infty) = \infty$, this gives $t\mathbb{P}_x(T \in (t, t+\Delta]) \backsim \Delta k_6 k_7 \tilde{h}_x(1)$, and this is easily seen to contradict the standard stable functional limit theorem unless $k_6k_7 = 1$. If $p = d^*/(d^* + L(\infty)) < 1$ we get $t\mathbb{P}_x(T \in (t, t+\Delta]) \backsim p^{-1}\Delta k_6 k_7 \tilde{h}_x(1)$ and the same argument gives $k_6k_7 = p$, and the results follow. ■

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